Jane Warner Cook

Alpha Teach Yourself

Algebra 1

in 24 hours

ALPHA A member of Penguin Group (USA) Inc.

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Introduction

At the start of every school year, my algebra students always ask me this crucial question:

"What does algebra have to do with real life?"

As you begin the journey in *Alpha Teach Yourself Algebra 1 in 24 Hours*, you may find yourself asking the same question. This book helps you find the answer. Along the way, you will discover that when you invest in algebra, you invest in yourself. Learning the hows and whys of algebra will expand your mind and open up new possibilities for your life.

Teaching yourself algebra is like taking the letters of the alphabet and putting them together to read a book. Basic arithmetic is algebra's alphabet. In the next 24 hours, you'll take what you already know and learn how to use it to "read" higher-level algebra.

One of the things you'll see early on is that you are already using algebra every day. This book simply helps you be more fluent in its language. Since you already have a good grasp of addition and multiplication from basic math, you have the foundation to take the next step up.

For example, each time you figure out which is the better deal, a 3-pound bag of potatoes for $1.99 or a 5-pound bag for $3.99, you're using algebra. Calculators and computers can do a lot of the math work, but only you can tell these machines what problem needs to be solved, which numbers should be used, and what needs to be done with the results.

Everyone works at a different rate. Some of the "hours" may take you less than 60 minutes, and others may take more. But no matter how much time elapses, over the course of the 24 chapters, you will see complex concepts broken down into manageable bits. You will find examples to follow and problems to try on your own, plus 10 problems at the end of each hour to reinforce what you've just learned.

Calculators are helpful for a few of the problems, but most of the calculations are done easily in your head or with pencil and paper. Most of the answers will be whole numbers because the goal is for you to focus on the algebraic concepts, not the math calculations.

Hold on tight. You are about to experience the joy of mastering a subject many people find difficult. It will make no difference if you've been away from algebra for years or if this is your first attempt. During the next 24 hours, new analytical skills will help you in all areas of your life. Algebra is not something everyone feels they can do. But you can. That's got to feel good.

Last but not least, this book has a lot of miscellaneous cross-references, tips, and shortcuts in sidebar boxes. Here's how they stack up:

FYI FYI sidebars offer advice or show an alternative approach for solving a problem.

TIME SAVER

Time Saver sidebars give you a faster way to do some of the more complex problems and often include shortcuts.

GO TO ▶

Go To sidebars give you cross-references to another chapter in the book.

STRICTLY DEFINED

Strictly Defined sidebars offer definitions of words you may not know.

PROCEED WITH CAUTION

Proceed with Caution sidebars warn you about potential problems and help you steer clear of trouble.

JUST A MINUTE

Just a Minute sidebars include important things you learned in prior chapters that are needed in the current one.

About the Author

Jane Warner Cook teaches algebra and geometry at Moravian Academy in Bethlehem, Pennsylvania. She has taught and tutored students in algebra I and II, geometry, and precalculus since 1972, when she graduated from Colorado Women's College with a degree in mathematics. She earned a Master's degree in secondary administration from Drake University in 1999. Ms. Cook presents complex algebraic concepts simply to self-learners in *Alpha Teach Yourself Algebra 1 in 24 Hours.*

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My gratitude extends to the many teachers and students I have worked with and learned from over the years. Mary Maxwell, my high school advanced algebra teacher, inspired me to become a math teacher. Phillip Rose, my calculus teacher in college, guided me patiently through concepts that I thought I would never understand. I am grateful to S. Leigh Nataro for the improvements she made in the manuscript.

Part I

The Language of Algebra

Hour 1 Algebraic Expressions

Hour 2 The Real Number Line

Hour 3 Real Numbers: Four Operations

Hour 4 Real Numbers: Properties and Rules

Hour 1

Algebraic Expressions

Chapter Summary

In this hour you will learn about …

- Recognizing symbols for multiplication and division
- Understanding grouping symbols
- Following the rules for order of operations
- Solving practical problems with algebraic expressions

Algebra is simply a language that lets you speak efficiently about numbers and how they relate. For example, when you first learned that "five plus two is seven," it was probably written like this:

$$\begin{array}{r} 5 \\ +2 \\ \hline 7 \end{array}$$

And when you first learned that "five times two is ten," it was probably written like this:

$$\begin{array}{r} 5 \\ \times 2 \\ \hline 10 \end{array}$$

A more efficient way to write these statements is: $5 + 2 = 7$ and $5 \cdot 2 = 10$. This is algebra at its simplest. What was a math problem has now become an algebraic sentence. It contains the most basic algebraic symbol, the equals sign. This small but mighty symbol has revolutionary power because it allows you to equate two things that really look quite different from each other. Writing and solving equations is probably the most important work you do in algebra.

GO TO ▶
You'll learn more about equations and how to solve them in Hour 5.

Recognizing New Symbols for Multiplication and Division

Learning some basic definitions in the language of algebra will give you the framework you need. The *terms* in the two original questions are 5 and 2, and the *operation* is adding in the first expression and multiplying in the second. The 5 and 2 are specific numbers. But in algebra, the terms can be *variables*, a key word based on "to vary" or "to change." If you change 5 + 2 = 7 (which you know is true) to $x + 2 = 7$, you've created an equation that may or may not be true, depending on the value of the variable x.

Many fundamental rules, definitions, and visual representations are the same in arithmetic and algebra, and others are unique to algebra. The + and – signs that show adding and subtracting in basic arithmetic are the same signs you use for adding and subtracting algebraic terms.

But be aware that multiplying terms in algebra can be indicated in a few different ways. To avoid any confusion by writing "7 times x" as $7 \times x$ (with the multiplication symbol looking a lot like the variable x), you'll usually see $7x$, $7 \cdot x$, or $(7)(x)$ in algebraic expressions. The division symbol, as in $a \div 12$, is the same in algebra and arithmetic, but division will sometimes appear as $a/12$ or, most often, as $\frac{a}{12}$.

A lot of the work in algebra involves *simplifying* long expressions that have many terms. Sometimes special symbols will guide you, and in other problems you'll need to know some rules. Fortunately, these symbols and rules are easy to recognize and use.

STRICTLY DEFINED

Simplifying expressions means using the grouping symbols to combine the terms using the operations of +, –, ×, and ÷ in the problem. The grouping symbols most commonly used are parentheses, ( ), or brackets, [ ].

Understanding Grouping Symbols

Let's begin with expressions that have only numbers and no variables. In 6 + (18 ÷ 2), you are trying to *combine* the numbers. The parentheses act like *grouping symbols* and tell you to first divide 18 by 2 and then add 6 to the result, which gives you a result of 15. In (6 + 18) ÷ 2, doing what's

in the parentheses first gives a result of 24, and dividing that by 2 gives you 12. This shows the importance of obeying the grouping symbols, because 15 and 24 are very different answers!

STRICTLY DEFINED

Combining terms means using the **grouping symbols** to **simplify** the problem. The grouping symbols most commonly used are parentheses, ( ), or brackets, [ ].

Now try these and check your answers on page 8:

1. Simplify: $8 + (6 \cdot 3)$
2. Simplify: $(8 + 6) \cdot 3$
3. Simplify: $2(6 - 2) + 4(7 - 5)$

Following the Rules for Order of Operations

Combining the numbers in problem 3 from the previous section will lead you to an expression without parentheses. Look at it again:

$2(6 - 2) + 4(7 - 5) = 2 \cdot 4 + 4 \cdot 2$

This situation requires knowing what to do when there is more than one operation and no parentheses. The correct order for what operation to do first is:

If there are no parentheses:

1. Multiply and divide first.
2. Then add and subtract.

For example, there are no parentheses in the following problem, so here is how to simplify it.

Simplify:

$12 + 16 \div 2 - 1 \cdot 3 =$
$12 + 8 - 3 =$
$20 - 3 =$
17

In this example, if you had just worked from left to right, doing the operations as they appear, you would get 39, which is not correct. Using the rules for order of operations, you must do the multiplying and dividing

before any adding or subtracting. So the 16 is divided by 2, and the 1 is multiplied by 3. The 12 and 8 are added, and the 3 is subtracted. If the problem includes many operations and also parentheses, do what's in the parentheses first.

Now try these problems and check your answers on page 8:

4. Simplify: $20 + 12 \div 2 - 4 \cdot 3$
5. Simplify: $15 \div (10 - 7) - 1 \cdot 4$
6. Simplify: $(4 + 3) - (8 - 5) + 16 \div 2 - 2 \cdot 3$

You have been using division of single numbers in many of the examples. Division in algebra problems is usually shown as a fraction and can often include more than one term in the *numerator* or *denominator*. When it is in this form and you are simplifying, first work on the numerator, then work on the denominator, and then divide the results.

STRICTLY DEFINED

Remember that the upper part of a fraction is called the **numerator** and the lower part is called the **denominator.**

For example:

Simplify: $\frac{8 + 4}{5 - 2} = \frac{12}{3} = 4$

Now try these problems and check your answers on page 8:

7. $\frac{9 - 5}{2 \times 2}$
8. $\frac{10 \cdot 3}{10 + 5}$

Now that you have practiced simplifying expressions that include different operations and grouping symbols, it's time to put those skills to work in problems that have variables.

Solving Practical Problems with Algebraic Expressions

An expression that includes variables can be evaluated, if you know the values for the variables. Follow the same rules for order of operations that you have already learned.

JUST A MINUTE

If there are parentheses, do what is inside them first. Then do the multiplying and dividing, as they appear from left to right. Then do the adding and subtracting, as they appear from left to right.

If there are no parentheses, multiply and divide first, from left to right, and then add and subtract, from left to right.

Evaluate: $6x - 7y + 2$, given $x = 4$, $y = 3$

Answer: Substitute the numbers for the variables.

$$6 \cdot 4 - 7 \cdot 3 + 2 = 24 - 21 + 2 = 5$$

Evaluate: $2x - 5$, given $x = 7$

Answer: $2 \cdot 7 - 5 = 14 - 5 = 9$

Evaluate: $2(x - 5)$, given $x = 7$

Answer: $2(7 - 5) = 2(2) = 4$

Now try these problems and check your answers on page 8:

9. Evaluate: $2a + 4 - 5b$, given $a = 6$ and $b = 2$
10. Evaluate: $12 - 6x - 2y + 5$, given $x = 1$ and $y = 3$

Several English phrases indicate addition, including *increased by*, *more than*, and *added to*. The following table lists these phrases, their operations, and their algebraic expressions.

English Word or Phrase	*Operation*	*Algebraic Expression*
sum of a and 3	Addition	$a + 3$
b more than 5	Addition	$5 + b$
c increased by 2	Addition	$c + 2$
6 decreased by x	Subtraction	$6 - x$
3 less than 8	Subtraction	$8 - 3$
y minus 1	Subtraction	$y - 1$
difference between x and y	Subtraction	$x - y$
6 times a	Multiplication	$6a$
product of 8 and x	Multiplication	$8x$
9 divided by y	Division	$9 \div y$ or $9/y$ or $\frac{9}{y}$
quotient of x and 4	Division	$x \div 4$ or $x/4$ or $\frac{x}{4}$

Referring to the previous table, write each of the following phrases as algebraic expressions. Use x as the variable. Check your answers below.

11. Five times a number
12. The sum of 2 and a number
13. Three times a number, decreased by eight
14. Twelve decreased by seven times a number
15. Eight times the sum of a number and 3
16. 9 plus the quotient of a number and 6

Answers to Sample Problems in Hour 1

1. $8 + (6 \cdot 3) = 8 + 18 = 26$
2. $(8 + 6) \cdot 3 = 42$
3. $2(6 - 2) + 4(7 - 5) = 2 \cdot 4 + 4 \cdot 2 = 8 + 8 = 16$ (If you had trouble with the third step in this solution, don't worry. It is explained in the section "Order of Operations") on page 5
4. $20 + 12 \div 2 - 4 \cdot 3 = 20 + 6 - 12 = 26 - 12 = 14$
5. $15 \div (10 - 7) - 1 \cdot 4 = 15 \div 3 - 4 = 5 - 4 = 1$
6. $(4 + 3) - (8 - 5) + 16 \div 2 - 2 \cdot 3 = 7 - 3 + 8 - 6 = 4 + 8 - 6 = 12 - 6 = 6$
7. $\dfrac{9 - 5}{2 \times 2} = \dfrac{4}{4} = 1$
8. $\dfrac{10 \cdot 3}{10 + 5} = \dfrac{30}{15} = 2$
9. $2a + 4 - 5b$, given $a = 6$ and $b = 2$; $2 \cdot 6 - 5 \cdot 2 = 12 + 4 - 10 = 6$
10. $12 - 6x - 2y + 5$, given $x = 1$ and $y = 3$; $12 - 6 \cdot 1 - 2 \cdot 3 + 5 = 12 - 6 - 6 + 5 = 5$
11. $5x$, $5 \cdot x$, or $(5)(x)$
12. $2 + x$
13. $3x - 8$
14. $12 - 7x$
15. $8(x + 3)$
16. $9 + x \div 6$ or $9 + x/6$ or $9 + \dfrac{x}{6}$

Hour's Up!

Review

Based on what you've learned in Hour 1, do the following problems about algebraic expressions. Check your answers with the solution key in Appendix B.

1. Simplify: $10 - (2 + 5) + (8 - 2)$
- **a.** 7
- **b.** 5
- **c.** 9
- **d.** 3

2. Simplify: $(10 - 2) + (5 + 8) - 2$
- **a.** 12
- **b.** 19
- **c.** 20
- **d.** 21

3. Simplify: $(2 + 7) - (6 - 1) + 9 \cdot 2 + 8 \div 4$
- **a.** 32
- **b.** 30
- **c.** 15
- **d.** 24

4. Simplify: $\dfrac{12 - 4}{2(8 - 6)}$
- **a.** 4
- **b.** 2
- **c.** 3
- **d.** 1

5. Simplify: $\dfrac{3(9 - 4)}{5(3 - 2)}$
- **a.** 3
- **b.** 5
- **c.** 7
- **d.** 9

6. Evaluate: $2a - 6b + 12 - c$, given $a = 3$, $b = 1$, and $c = 10$
- **a.** 8
- **b.** 5
- **c.** 2
- **d.** 1

Review

7. Evaluate: $9 + (x - 3) - (12 - y) - 5$, given $x = 5$ and $y = 7$

a. 1
b. 8
c. 4
d. 2

Write the following phrases as algebraic expressions:

8. 6 increased by the product of x and 3

a. $6 + 3 \div x$
b. $6 + 3x$
c. $6 + 3 - x$
d. $6 - 3x$

9. 5 subtracted from the sum of a and y

a. $5 - (a - y)$
b. $5 - (a + y)$
c. $(a + y) - 5$
d. $(a - y) - 5$

10. c decreased by the sum of 4 and d

a. $c - (4 - d)$
b. $c - (4 + d)$
c. $(4 - d) - c$
d. $(4 + d) - c$

Hour 2

The Real Number Line

CHAPTER SUMMARY

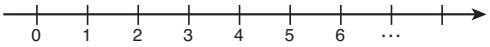
In this hour you will learn about …

- Plotting positive and negative numbers and zero
- Ordering rational and irrational numbers
- Working with real numbers
- Finding absolute values

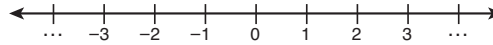
In Hour 1, you built on your knowledge of basic arithmetic and learned how to evaluate expressions that include variables. In this hour, you move beyond the numbers you're so familiar with, widening your circle of algebraic knowledge.

Plotting Positives, Negatives, and Zero

The numbers you use for many computation problems are 0, 1, 2, 3, and so on. In algebra, these are defined as *whole numbers*, and they continue without an end. There is always at least one more whole number after the one you are thinking of at the time. Picture a simple ruler or measuring tape that has only whole numbers. It might look like this one, showing the whole numbers up to 6:



But in real life, you need negative numbers, too, such as when your checking account is overdrawn, for example, or when it is 10 degrees below zero outside. Picture a number line that doesn't begin at zero but has the equal intervals also going to the left of zero. It would look like this, with the numbers increasing to the right and decreasing to the left.



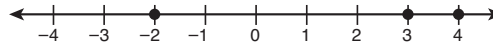
The algebraic term for the negative and positive whole numbers and zero, as shown on the number line, is *integers*. Another way to describe integers is all the whole numbers along with their *opposites*, and zero. The set of integers is *infinite*, which means it has no beginning or end, since there is always at least one integer smaller than the smallest one or at least one larger than the largest one you are considering.

STRICTLY DEFINED

The **opposite** of a number is the number with a different sign. The opposite of 2 is –2. The opposite of –6 is 6.

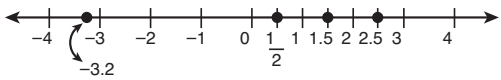
In algebra books, the positive integers are often listed as {1, 2, 3, …} and the negative integers as {–1, –2, –3, …}, with the three dots showing that the numbers continue without an end. Notice that the positive integers don't always have the + sign. If no sign is shown, assume that the number is positive.

Now consider *plotting* (this is another way of saying graphing) an integer. Find the number's place on the number line and mark it (plot it). For example, let's plot the following integers on the number line: –2, +4, and 3.



Consecutive integers are numbers that are next to each other on the number line. Given any integer, the consecutive integer is found by moving one interval to the right on the number line. For example, 3 and 4 are consecutive integers, as are –2 and –1.

So far, the focus has been on integers. In real life, though, you also need fractions and decimals. You could think of fractions and decimals as numbers that are between the integers on the number line. This means that they can also be plotted. In fact, every integer (let's use 5 as an example) can also be written as a fraction (as $\frac{5}{1}$) or as a decimal (as 5.0). Look at the number line that follows to see where 2.5, –3.2, +1.5, and $\frac{1}{2}$ are plotted.



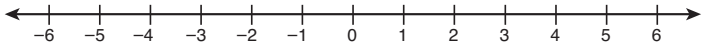
Any number between –4 and +4—whether it's an integer, fraction, or decimal—has a unique place on this number line. This fact is one of the fundamental properties of algebra. In algebra, all the numbers you've learned about in this hour— the whole numbers, integers, fractions, and decimals—are called *rational numbers* because they all can be written as the *ratio* of two integers, as a fraction.

STRICTLY DEFINED

If a number can be written as the "ratio" of two integers, it's defined as a **rational number.** Every integer, fraction, and decimal can be written as a ratio of two integers, so they are all called rational numbers.

Now try this problem and check your answers on page 18.

1. Plot the following numbers on the number line: 4, –2, 5.5, $-\frac{1}{2}$, and –4.2



Numbers that cannot be written as a ratio of two integers include square roots, such as $\sqrt{2}$. These are called irrational numbers, and you'll learn about them in Hour 21, including how to plot them on a number line and how to use them algebraically. Irrational numbers are mentioned here because, when thought of together, the rational and irrational numbers are called the real numbers; therefore, they can all be plotted on the real number line.

Ordering Rational and Irrational Numbers

A number line helps you visualize where a number is in relation to another number. Because the number line has smaller numbers on the left side and larger numbers on the right, you can compare plotted numbers to see which is greater than the other. Numbers can be *ordered* (compared to each other) with inequality symbols, such as greater than (>) or less than (<). Of course, if the numbers are equal, you use the =

symbol. Just as *equal to* is shown with the = symbol, *greater than* and *less than* have their own symbols.

Symbols for Ordering Numbers

Phrase	*Symbol*
"is greater than"	>
"is less than"	<
"is equal to"	=
"is greater than or equal to"	≥
"is less than or equal to"	≤

FYI If you have trouble remembering which symbol represents "greater than" and which represents "less than," it may help to picture the "less than" symbol < as an *L,* and "less than" starts with an *L.*

The combined statement "4 is greater than or equal to 3" (written $4 \geq 3$) allows for either $4 > 3$ *or* $4 = 3$, which is true because 4 is clearly greater than 3. The combined statement "2 is less than or equal to 2" (written $2 \leq 2$) is also true, since 2 does equal 2.

You need to be able to use these four symbols (<, >, ≤, and ≥) to compare numbers now and also in Hour 19. Let's use them in a few examples:

a. Say: "2 is less than 5." Write: $2 < 5$
b. Say: "4 is greater than 0." Write: $4 > 0$
c. Say: "5 is greater than –1." Write: $5 > -1$
d. Say: "–3 is less than or equal to 3." Write: $-3 \leq 3$
e. Say: "1.5 is greater than or equal to 1.5." Write: $1.5 \geq 1.5$
f. Say: "–1.5 is less than or equal to 2." Write: $-1.5 \leq 2$

Now try these problems and check your answers on page 18.

Write the following expressions in algebraic form:

2. –8 is greater than or equal to –9
3. 0 is less than or equal to 5
4. 5 is greater than 0
5. 10 is greater than –10

Write the following algebraic expressions in words:

6. $12 \leq 14$
7. $-6 > -10$
8. $-5 \geq -8$
9. $3 \leq 3$

Ordering numbers means you use the inequality symbols you just learned (< and >) to correctly write algebraic sentences. Is 4 less than, greater than, or equal to 1? Because 4 is less than 1, 4 < 1 makes a true statement. Fill in the blanks on these inequalities with < or >:

g. –5 ___ 9
   Answer: (–5 is less than 9, so use <)
h. –8 ___ 0
   Answer: (–8 is less than 0, so use <)
i. –2 ___ –6
   Answer: (–2 is greater than -6, so use >).

Now that you can compare two given numbers and write a sentence with the appropriate inequality symbol, think of several numbers and how they can be ordered from least to greatest. It may help you to plot them on a real number line.

STRICTLY DEFINED

Ordering numbers can involve comparing two numbers and using the correct inequality symbol between them, as in –1 < 1, or it can involve listing a group of numbers in ascending order, as in –3, –1, 0, 2.5, 5. Plotting the numbers on a real number line will help you see the correct order, increasing from left to right.

Consider the following list of integers:

9, 6, –4, 2, 0, –5

Because –5 is the least of these (it's farthest to the left on the number line), it will be first in the list, followed by –4. The remaining positive integers and zero just need to be put in ascending order. So the correct sequence from least to greatest is:

–5, –4, 0, 2, 6, 9

Working with Real Numbers

Including fractions and decimals in this sequencing or ordering process involves comparing them to their nearest integer values.

Consider this list of numbers:

$$3, -2.5, \frac{3}{4}, 8, -3, 0$$

Begin with the negative numbers, –2.5 and –3, because they are clearly the smallest numbers. –2.5 is between –2 and –3 on the number line, with –2 to the *right* of –3. Because –2.5 is also to the *right* of –3, you could write –2.5 > –3. From this process, you can see that the first number in the order is –3, followed by –2.5. The remaining numbers are easily put in ascending order, so the correct sequence is:

$$-3, -2.5, 0, \frac{3}{4}, 3, 8$$

Now try these problems and check your answers on page 18.

Fill in the blanks of the following inequalities with < or >:

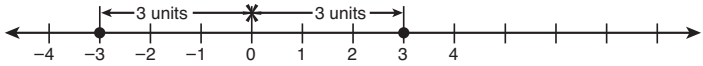
10. –6 ___ 8
11. 6 ___ –8
12. –6 ___ –8
13. 7 ___ –7
14. .5 ___ 1
15. $\frac{1}{4}$ ___ 2
16. $-\frac{1}{4}$ ___ –2

Write the following numbers in order from least to greatest:

17. 4, –5, .3, 7, –.5, 0, 3
18. 6, 0, –3, –3.1, 8, $-\frac{1}{2}$

Finding Absolute Values

To prepare for Hour 3, which includes adding, subtracting, multiplying, and dividing positive and negative numbers, look again at the number line.



+3 is three units to the right of 0. –3 is also three units away from 0, but to the left. You already know that +3 > –3 and that –3 < +3. But because they both are the same distance from 0, they have the same *absolute value*. A number's absolute value is defined to be its distance from 0. Distance is not considered to be positive or negative, so a number's absolute value has no + or – sign in front. The absolute value symbol is | |, with the number inside the vertical lines.

STRICTLY DEFINED

A number's **absolute value** tells how far it is from zero on the number line. To simplify an absolute value expression, just remove the number's sign, because distance in algebra is not positive or negative; it's neutral. The absolute value of –1 is 1, and this can be written as |–1| = 1.

When you see |–5| = 5, you say, "The absolute value of negative five equals five." Here are some examples:

j. $|-1| = 1$
k. $|+4| = 4$
l. $|-4| = 4$
m. $\left|-\frac{1}{2}\right| = \frac{1}{2}$

You will appreciate the need for finding a number's absolute value in Hour 3, but for now, add it to your algebra toolbox.

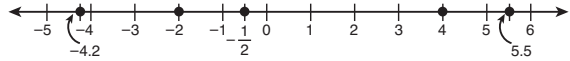
Now try these problems and check your answers on page 18–19. (Problems 19 through 25 include absolute value terms along with a review of the order of operations rules you learned in Hour 1. They are very important in Hour 3!)

19. $|-2| =$ ___
20. $|+8| =$ ___
21. $|-.5| =$ ___
22. $|+.5| =$ ___
23. $12 \div 2 - 3 \cdot 2 + (|-4| + 6) - 5 =$
24. $(3 - |-1|) + 10 - 2 \cdot 4 =$
25. $14 - 3 \cdot 3 - 2 + |+8| =$

Before you work on the problems in the "Hour's Up!" section, let's review the types of numbers you've learned about so far.

Whole numbers = {0, 1, 2, 3, . . .}
Integers = { … , –2, –1, 0, 1, 2, 3, …}
Rational numbers can be written as the ratio of two integers.
Irrational numbers cannot be written as the ratio of two integers (more about them in Hour 21).
Real numbers are any number that is rational or irrational (basically, any number).
A number's absolute value is its distance from 0.

Answers to Sample Problems in Hour 2

1. 

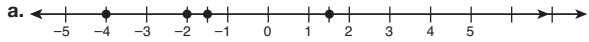
2. $-8 \geq -9$
3. $0 \leq 5$
4. $5 > 0$
5. $10 > -10$
6. 12 is less than or equal to 14
7. –6 is greater than –10
8. –5 is greater than or equal to –8
9. 3 is less than or equal to 3
10. $-6 < 8$
11. $6 > -8$
12. $-6 > -8$
13. $7 > -7$
14. $.5 < 1$
15. $\frac{1}{4} < 2$
16. $-\frac{1}{4} > -2$
17. –5,–.5,0,.3,3,4,7
18. $-3.1, -3, -\frac{1}{2}, 0, 6, 8$
19. 2
20. 8
21. .5

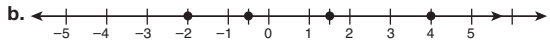
22. .5
23. $12 \div 2 - 3 \cdot 2 + (|-4| + 6) - 5 =$
$12 \div 2 - 3 \cdot 2 + (4 + 6) - 5 =$
$12 \div 2 - 3 \cdot 2 + (10) - 5 =$
$6 - 6 + 10 - 5 =$
$0 + 10 - 5 =$
$10 - 5 = 5$
24. $(3 - |-1|) + 10 - 2 \cdot 4 =$
$(3 - 1) + 10 - 2 \cdot 4 =$
$(2) + 10 - 2 \cdot 4 =$
$2 + 10 - 8 =$
$12 - 8 = 4$
25. $14 - 3 \cdot 3 - 2 + |+8| =$
$14 - 9 - 2 + 8 =$
$5 - 2 + 8 =$
$3 + 8 = 11$

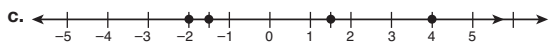
Hour's Up!

Based on what you've learned in Hours 1 and 2, do the following problems about algebraic expressions. Check your answers with the solution key in Appendix B.

1. Plot 4, –2, 1.5, and –1.5 on the real number line.

a.


b.


c.


2. Write "7 is greater than –8" in algebraic form.
a. $-7 > 8$
b. $-7 < 8$
c. $7 > -8$
d. $7 < 8$

3. Write "–3 is less than or equal to –2.5" in algebraic form.
a. $-3 \leq 2.5$
b. $-3 \leq -2.5$
c. $3 \leq 2.5$
d. $-3 \geq 2.5$

Review

4. Write $12 \leq 15$ in statement form.
a. 12 is less than 15
b. –12 is less than 15
c. 12 is greater than or equal to 15
d. 12 is less than or equal to 15

5. Write $-4 > -15$ in statement form.
a. –4 is greater than or equal to 15
b. –4 is greater than or equal to –15
c. –4 is less than –15
d. –4 is greater than –15

6. Use < or > to fill in the inequality –4 ___ –7.
a. <
b. >

7. Use < or > to fill in the inequality 0 ___ –3.
a. <
b. >

8. Order 8, –4, 4, –2, 0 from least to greatest.
a. 8, 4, 0, –2, –4
b. 8, 4, –2, –4, 0
c. 0, –2, –4, 4, 8
d. –4, –2, 0, 4, 8

9. $|-6| =$
a. –6
b. 6

10. Simplify $7 + (5 - 2) - (|-12| - 4) \div (8 - 6)$
a. 6
b. 0
c. 3
d. 4

Review

Hour 3

Real Numbers: Four Operations

CHAPTER SUMMARY

In this hour you will learn about …

- Adding, subtracting, multiplying, and dividing real numbers

In Hours 1 and 2, your math world expanded beyond the whole numbers to include positive and negative integers, real numbers, inequalities, and absolute value. Now, to use this knowledge, you learn the rules for adding, subtracting, multiplying, and dividing real numbers. These four operations with real numbers form an important foundation that allows you to set up and solve equations and advance to higher algebra skills.

Adding Real Numbers

Adding two whole numbers is probably one of the first math problems you learned to solve, right after you learned to count from 1 to 20. Because algebraic expressions include all the real numbers, not just whole numbers, you also need to be able to add real numbers.

Even though your understanding of numbers has grown quite a bit in the past two hours of study, the basics remain the same. It's still true that $2 + 3 = 5$ and that $10 + 6 = 16$. As long as an algebra problem has only whole numbers, you will be fine using your basic math facts. Many algebra problems do include negative numbers, and fortunately, some basic rules will guide you. But before we get to the rules, it might be helpful to know when you will use them.

You'll see negative signs on some of the terms in the next examples. As you reach the end of this section, you will be able to simplify each of these problems:

$$-1 + -10 =$$
$$9 + -7 =$$
$$-2 + -8 + -11 =$$
$$3 + -9 + 14 =$$

These examples are addition problems involving some negative and some positive numbers.

Look closely at the first example: $-1 + -10 =$.

The signs can be confusing, but in words, you would say "negative 1 plus negative 10." This helps you realize that you are trying to add a negative number to another negative number.

A real-life example of this situation is when you owe someone $1 and then borrow another $10. Now you owe this person a total of $11. But because it's a debt, it's really –11. You can see clearly that adding –1 and –10 will give a sum of –11.

Let's get back to the algebra. The rule is that, when adding negative numbers, you remove their negative signs, add them, and put the negative sign on the final answer. Mathematically, adding two or more negative numbers gives a negative number.

FYI In algebra problems, parentheses surrounding more than one term affect the order of operations (as described in Hour 1), but when used around a single term, they exist only to separate or emphasize the terms and are not used as grouping symbols. $-1 + -10$ is the same as $(-1) + (-10)$.

The next four problems show the different ways you will see parentheses in algebra problems.

a. $-4 + -2 = (-6)$
b. $-4 + (-2) = -6$
c. $(-4) + (-2) = -6$
d. $(-4) + -2 = -6$

You probably noticed that each of these problems is a different version of the same problem, which, at its simplest, is –4 plus –2. The parentheses do not show multiplication, which, you will recall from Hour 1, can sometimes be the case. They are also not grouping symbols. They are simply a form of emphasis and may or may not appear in a given problem.

Being aware that the parentheses have no power when they surround a single term frees you to focus on the operation itself, which, in this case, is just adding two negative numbers. Because you will see parentheses in some problems and not in others, be prepared to determine whether they are showing multiplication, as they did in Hour 1, or are just separating the terms, as in these examples. You will see some examples of parentheses used for multiplying real numbers later this hour, on page 29. Remember that if the numbers being added are positive, the sign of the answer will be positive. If the signs of all the numbers being added are negative, the answer will be negative.

Now try these six problems and check your answers on page 30.

1. $-6 + -7 =$
2. $(-5) + -2 =$
3. $(-1) + (-9) =$

The problems that follow have more than two negative numbers, but the same rule applies: remove their negative signs, add them, and put the negative sign on the final answer.

4. $-9 + -1 + -3 =$
5. $(-3) + -2 + (-1) =$
6. $-5 + (-4) + (-10) =$

Now consider the following problem:

$$9 + -7 =$$

This is an addition problem; in words, you would say "9 plus negative 7." The rule you just learned will work perfectly if all the numbers are positive or all are negative. But one of these numbers is positive and the other is negative, so the rule doesn't apply here.

The problem is actually 9 plus the opposite of 7. In effect, then, this is a subtraction problem. The key concept here is that adding a negative number is the same as adding the opposite of a positive number.

The formal rule for adding two numbers with different signs is to subtract their absolute values and then use the sign of the larger absolute value for the sign in the answer. More simply, you remove the signs from the numbers, subtract them, and use the larger number's sign for the answer.

Look again at the example in this section: 9 + –7 = ___.

Using this rule, you think 9 – 7 = 2, and because 9 is larger and positive, the answer is +2, which you usually write as 2.

Look at some more examples and solutions of adding numbers with different signs:

e. –4 + 7 = 3
f. 3 + –8 = –5
g. –10 + 3 = –7
h. (–5) + (+ 6) = 1
i. 2 + (–9) = –7

Now try the following problems, and check your answers on pages 30–31. (Watch the signs carefully to know which rule you will use.)

7. –7 + 3 =
8. (–5) + (–8) =
9. 12 + (–2) =
10. (–1) + 6 =

To review what you've learned so far about adding real numbers:

> Adding all positive numbers gives a positive number.
> Adding all negative numbers gives a negative number.
> To add a positive to a negative number, remove their signs, subtract, and use the sign of the larger number for the sign of the answer.

You are probably wondering what to do if there is a mixture of positive and negative numbers within an addition problem. Once you've mastered the previous rules, having several numbers to add, no matter what their signs, is not difficult.

Consider the following problem:

3 + –9 + 14 =

The 3 and 14 are both positive, and the 9 is negative. You have two choices here. The first is to work the problem from left to right, as you read it. What is 3 + –9? Since the signs are different, remove the signs, subtract the numbers, and use the sign of the larger number. So 3 + –9 = –6. To continue the problem, you need to ask, what is –6 + 14? Remove the signs, subtract the numbers, and use the sign of the larger number for the answer. Since –6 + 14 = 8, this is the final answer. (See the FYI note for another way to do this problem.)

FYI Another way to simplify 3 + –9 + 14 is to group the positive numbers together, find their sum, and then deal with the negative term. Grouping the positives gives 3 + 14 = 17. The next step would be 17 + –9, which is 8. You are free to choose the method that works most easily for you.

The following examples include both positive and negative terms:

j. 1 + –3 + 6 = 4
k. –2 + –2 + 5 = 1
l. (–6) + –3 + 9 = 0
m. (–5) + 7 + (–4) = –2
n. 3 + –8 + –1 + 2 = –4

Now try these problems and check your answers on page 31.

11. –6 + 8 + –1 =
12. (–7) + 5 + (–3) + (–2) =
13. (–10) + 4 + (–9) + 5 =

Subtracting Real Numbers

Now that you have learned how to use the rules for adding real numbers, you are ready to shift gears and try the following basic subtraction problems:

9 – 6 =
7 – 2 =
5 – 0 =
8 – 4 – 2 =

These simple problems are easily solved. Because they are all examples of subtracting a smaller whole number from a larger one, they don't require any higher skills than basic arithmetic. (Did you get 3, 5, 5, and 2 for the answers?)

Now consider these more complex subtraction problems:

$$8 - (-2) =$$
$$-10 - (-3) =$$
$$-7 - 4 =$$
$$5 - (-6) =$$

These subtraction problems include positive and negative integers, so they are not as easy as the first four. To learn how to solve them, you need to first see the close link between the operations of addition and subtraction.

You will see this link most clearly if you begin by looking back at a problem that involves adding positive and negative integers, such as this one:

$$10 + (-6) =$$

Because you are adding and the signs are different, you know the rule: remove the signs, subtract the numbers, and use the sign of the larger number for the sign of the answer.

So you know that $10 + (-6) = 4$.

Now look at the simple subtraction problem $10 - 6 = 4$. The answer to both of these problems is clearly 4. Is it a coincidence that you got the same answer? No, because adding –6 to 10 is exactly the same as subtracting 6 from 10, so, of course, the answers are the same.

The close link between subtraction and addition is most easily stated this way: subtracting a number is the same as adding a number with the opposite sign. To see this for yourself, consider these subtraction-to-addition examples:

$5 - 3 = 5 + (-3)$
Subtracting 3 is the same as adding –3.

$10 - 7 = 10 + (-7)$
Subtracting 7 is the same as adding –7.

$4 - 9 = 4 + (-9)$
Subtracting 9 is the same as adding –9.

(Did you get 2, 3, and –5 for your answers?)

JUST A MINUTE

The opposite of a number is the number with a different sign. The opposite of 2 is –2. The opposite of –6 is 6.

In the previous three problems, the first term in each problem is a positive number, but the same rule applies if the first term is negative:

–6 – 2 = –6 + (–2)
Subtracting 2 is the same as adding –2.

–1 – 5 = –1 + (–5)
Subtracting 5 is the same as adding –5.

(Did you get –8 and –6?)

In fact, all subtraction problems can be changed to addition, as long as you change the sign of the second term. The beauty of this process is that you can always get the answer to any addition problem because you know all the addition rules. You don't need any subtraction rules! Look at some samples:

o. 6 – (–4) = 6 + 4 = 10
p. –5 – (–3) = –5 + 3 = –2
q. 1 – 9 = 1 + (–9) = –8
r. 10 – 14 = 10 + (–14) = –4

To review, when you have a subtraction problem, change it to addition and change the sign on the second number.

JUST A MINUTE

Parentheses used around single terms do not show multiplication when there are addition or subtraction signs. They are just separating the operation (either adding or subtracting) from the signs of the numbers being used in the problem.

Now try these problems and check your answers on page 31.

14. –3 – (–5) =
15. –2 – 7 =
16. 10 – (–8) =

If a problem has many positive and negative numbers to be added and subtracted, you should change the subtraction to addition first and then

follow the rules for adding positive and negative numbers. Here are some examples, including each step for finding the final solution:

q. $9 + 5 - (-3) + 8 - (-4) =$
$9 + 5 + 3 + 8 + 4 =$
$14 + 3 + 8 + 4 =$
$17 + 8 + 4 =$
$25 + 4 = 29$

In the previous example, all the subtraction became addition of positive terms, so it is actually an easy problem to solve.

r. $-6 + 2 - (-8) - 4 + 7 =$
$-6 + 2 + 8 + (-4) + 7 =$
$-4 + 8 + (-4) + 7 =$
$4 + (-4) + 7 =$
$0 + 7 = 7$

In the previous example, the steps of the solution show the addition done from left to right. You could also group the positives together and get a total, group the negatives together and get a total, and then combine the positive and negative totals. The final answer will still be 7, using either method.

Now try these problems and check your answers on page 31.

PROCEED WITH CAUTION

Parentheses and brackets can be used as grouping symbols, and the order of operations learned in Hour 1 requires you to do the operations inside the parentheses first. This tip applies to problems 19 and 20.

17. $5 - 7 + 6 + (-3) - 1 - (-8) =$
18. $6 + (-3) - 1 - 5 - (-10) =$
19. $5 + (2 - 6) - (8 - 5) + [7 - (2 + 6)] =$
20. $-1 + 6 - (-2) - (4 + 3) + [(7 - 9) - 6] =$

Multiplying Real Numbers

Now that you have mastered adding and subtracting real numbers, the most difficult part of this hour is behind you. The rules for multiplying (and dividing) are simpler and very easy to use. In fact, let's go straight to the rules.

Multiplying positive numbers always gives a positive answer. This is true no matter how many positive numbers are in the problem. Examples:

$3 \cdot (+2) \cdot 4 = 24$ (The parentheses just emphasize the positive 2.)

$(+5)(6) = 30$ (The parentheses show multiplication.)

Multiplying two negative numbers always gives a positive answer. Yes, this seems strange, but it's true. Just be careful not to confuse the multiplying rule with the adding rule, since adding two or more negative terms gives a negative answer.

Examples:

s. $(-2) \cdot (-5) = 10$ (Multiplying two negatives → positive answer)
t. $-8 \cdot -2 = 16$ (Multiplying two negatives → positive answer)
u. $-8 + -2 = -10$ (Adding two negatives → negative answer)

Multiplying a positive number times a negative number always gives a negative answer. It doesn't matter which is larger or smaller; multiplying numbers with two different signs always gives a negative product.

Examples:

v. $6 \cdot -2 = -12$
w. $-6 \cdot 2 = -12$

When there are more than two numbers in the problem, apply the three multiplication rules as needed. The following examples show the solution steps:

x. $(-3) \cdot (2)(-4) = 24$ (Since $-3 \cdot 2 = -6$, the problem becomes $-6 \cdot -4 = 24$.)
y. $-2 \cdot -6 \cdot -1 = -12$ (Since $-2 \cdot -6 = 12$, the problem becomes $12 \cdot -1 = -12$.)

TIME SAVER

Another way to find the sign of the final answer in a multiplication problem is to count the number of negative numbers in the problem. Since two negatives multiply to give a positive, an even number of negative numbers gives a final answer that's positive. An odd number of negative numbers gives a negative answer. If you forget the even/odd rule, you can always use the method shown in examples x and y.

Now try these problems and check your answers on page 31.

21. $(-8) \cdot (2) =$
22. $(-6) \cdot (-3) =$

23. $5 \cdot (-2) \cdot (-3) =$
24. $(-5) \cdot 2 \cdot (-3) \cdot (-1) =$

Dividing Real Numbers

Knowing the rules for multiplying positive and negative numbers helps you know how to divide positive and negative numbers. In fact, the rules are exactly the same for both operations. Here are the division rules:

> Dividing two positive numbers gives a positive answer.
>
> Dividing two negative numbers gives a positive answer.
>
> Dividing two numbers with different signs gives a negative answer.
>
> When there are more than two numbers in a division problem, follow the same steps you use for finding the sign in multiplication.
>
> Also, if multiplication and division both appear in a problem, the same rules apply as if only one operation appears.

Examples follow:

z. $14 \div 2 = 7$
aa. $-14 \div -2 = 7$
bb. $(24) \div (-2) \div (-3) = (-12) \div (-3) = 4$
cc. $18 \div (-3) \cdot (-2) = -6 \cdot -2 = 12$

Now try these problems and check your answers on page 31.

25. $-25 \cdot -5 =$
26. $(-10) \div -2 \div (-1) =$
27. $-24 \div -2 \div 3 \cdot (-8) =$

Answers to Sample Problems in Hour 3

1. –13
2. –7
3. –10
4. –13
5. –6
6. –19
7. –4
8. –13
9. 10

10. 5
11. 1
12. –7
13. –10
14. 2
15. –9
16. 18
17. 8
18. 7
19. –3
20. –8
21. –16
22. 18
23. 30
24. –30
25. 5
26. –5
27. –32

Hour's Up!

Based on what you've learned in Hour 3, solve the following about adding, subtracting, multiplying, and dividing real numbers. Check your answers with the solution key in Appendix B.

1. –5 + –2 =
- **a.** 3
- **b.** +7
- **c.** –7
- **d.** –3

2. 6 + (–6) =
- **a.** 0
- **b.** –1
- **c.** –6
- **d.** 6

3. –3 + –10 =
- **a.** –7
- **b.** +7
- **c.** +13
- **d.** –13

Review

4. $7 - (-2) =$
a. 5
b. –5
c. 9
d. –9

5. $-8 - (-2) =$
a. –6
b. +6
c. –10
d. +10

6. $3 + (-5) + 6 + (-1) =$
a. 15
b. +3
c. –2
d. 0

7. $(-2) - 9 + (-3 + 6) - 8 =$
a. 0
b. –11
c. –16
d. +16

8. $5 \cdot -3 =$
a. 2
b. –2
c. +15
d. –15

9. $(-4) \cdot (-2) \cdot 3 \cdot (-1) =$
a. +24
b. –24
c. +12
d. –12

10. $(-18) \div 6 \cdot (-4) \div 3 =$
a. 8
b. –4
c. +4
d. –8

Review

Hour 4

Real Numbers: Properties and Rules

CHAPTER SUMMARY

In this hour you will learn about …

- Identifying the properties of real numbers
- Using the properties of real numbers
- Using the multiplication rule with real numbers
- Using the division rule with real numbers

Now that you know the basic operations of adding, subtracting, multiplying, and dividing positive and negative numbers, in this hour you learn the algebraic *properties* or rules that go beyond the four operations. These properties lay the groundwork for working with algebraic expressions—from the simple to the more complex.

Identifying the Properties of Real Numbers

Algebraic properties are much like the rules you learn for playing a sport such as football or basketball. These games would be chaotic without an agreed-upon set of standards, and it's the same for algebra.

Properties lay a strong foundation and give algebra its consistency and predictability. Unlike the English language, with so many exceptions to most of its rules of grammar and usage, the algebraic properties provide a firm base on which to rely. Solving an algebraic equation, a process you will learn in Hour 5, requires knowing how to use one or more of these properties.

Some of the properties are so basic that they seem unnecessary, and others are more complex, but each plays an important role in algebra. Let's begin with seven of the basic arithmetic properties, each followed by the same property in algebraic form.

Commutative Property

In basic arithmetic, you can add or multiply two numbers in any order, and the answer will be the same. For example, $2 + 3 = 3 + 2$ and $2 \cdot 3 = 3 \cdot 2$.

In algebra, you can add or multiply two algebraic values in any order, and the answer will be the same. For example, $a + b = b + a$ and $a \cdot b = b \cdot a$.

Associative Property

In basic arithmetic, you can add or multiply three or more numbers by grouping them in any order, and the answer will be the same. For example, $(2 + 3) + 4 = 2 + (3 + 4)$ and $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$.

In algebra, you can add or multiply three or more algebraic values by grouping them in any order, and the answer will be the same. For example, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Distributive Property

In basic arithmetic, multiplying a sum by a number is the same as adding the product of the numbers. For example, $4 \cdot (2 + 3) = (4 \cdot 2) + (4 \cdot 3)$.

In algebra, multiplying a sum by a number is the same as adding the product of the numbers. For example, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Identity Property of Addition

In basic arithmetic, if you add 0 to any number, the answer is the number itself. For example, $6 + 0 = 6$.

In algebra, if you add 0 to any term, the answer is the term itself. For example, $a + 0 = a$.

Identity Property of Multiplication

In basic arithmetic, if you multiply any number by 1, the answer is the number itself. For example, $3 \cdot 1 = 3$.

In algebra, if you multiply any value by 1, the answer is the number itself. For example, $a \cdot 1 = a$.

Opposites Property of Addition

In basic arithmetic, if you add any number and its opposite, the answer will be 0. For example, $10 + (-10) = 0$.

In algebra, if you add any term and its opposite, the answer will be 0. For example, $a + (-a) = 0$.

Reciprocal Property of Multiplication

In basic arithmetic, if you multiply any number by its *reciprocal*, the answer will be 1. For example, $4 \cdot \frac{1}{4} = 1$.

In algebra, if you multiply any number by its reciprocal, the answer will be 1. For example, $a \cdot \frac{1}{a} = 1$.

STRICTLY DEFINED

Reciprocals are two numbers that, when multiplied, equal 1, such as $2 \cdot \frac{1}{2} = 1$. To find a number's reciprocal, switch the numbers in the numerator and denominator. The reciprocal of $\frac{7}{8}$ is $\frac{8}{7}$; the reciprocal of -6 is $-\frac{1}{6}$. All real numbers except 0 have a reciprocal.

These seven properties are very important elements that will reappear throughout your remaining hours of learning algebra. You may never need to refer to them by name, but you must be able to use them with ease as you make the transition from basic arithmetic to algebra.

Using the Properties of Real Numbers

You already used some of these properties in Hour 3 when you learned how to add positive and negative numbers. Look back at an example from that section:

$$9 + (-2) + (-1) =$$

To find the answer, which is 6, you probably first added -2 to -1, getting -3. You then combined that with 9 to get your final answer, 6. You were actually using the associative property of addition when you chose to add the two negative terms first and then added the answer to 9.

Or maybe you combined the 9 and –2 first, getting 7, and combined that 7 with –1 to get the correct answer, 6. Grouping the addition terms in the order of your own choice is made possible by the associative property.

So at this basic level, the properties might not appear to be important; however, they do play a necessary role when it comes to *simplifying* expressions that include numbers and variables. Consider this problem:

> Simplify $3(x + 2)$.

STRICTLY DEFINED

To **simplify** means to write an algebraic expression in the fewest terms possible.

To state the problem in words, you could say, "Multiply 3 times the sum of x plus 2." The order of operations tells you to add the x and the 2 first because they are inside the parentheses, but they cannot be added together since one is a variable and one is a number. However, the distributive property allows you to change the problem to be "Multiply 3 times x, multiply 3 times 2, and then add the results."

So the way to simplify $3(x + 2)$ is to write $3x + 6$.

This specific example points out an important fact of algebra:

> Expressions that include parentheses are not in simplest form if the parentheses can be removed using the distributive property.

As you solve each problem with the distributive property, you can go directly from the problem to the answer without going through all the steps just described. The problem and its answer would look like this:

Simplify $3(x + 2) = 3x + 6$.

This brings up an important question: When is the answer to an algebra problem really in simplest form? You are finished simplifying when the answer has a single term or when all the *like terms* in the answer have been *combined*.

In the answer $3x + 6$, there are two terms, $3x$ and 6, and they cannot be combined (in this case, added together) because they are *unlike;* that is, one has a variable and one does not.

An example of like terms is $3x + 6x$, which can be simplified to equal $9x$. This combination of like terms is a perfect example of the distributive property in reverse order, because the sum of $3x + 6x$ could have come from $x(3 + 6)$. You know that $3 + 6 = 9$, which leads to $x \cdot 9$, which equals $9x$ because of the commutative property.

JUST A MINUTE

Recall the sign rules for + and ·. Adding positive numbers gives a positive number. Adding negative numbers gives a negative number. To add a positive to a negative number, remove their signs, subtract, and use the sign of the larger number for the sign of the answer.

Multiplying two positive numbers gives a positive answer.

Multiplying two negative numbers also gives a positive answer.

Multiplying two numbers with different signs gives a negative answer.

Seldom, if ever, will you need to go through all these steps to get from the problem of simplifying $3(x + 2)$ to the answer, which is $3x + 6$. But the detail you just followed in this section shows you the power of the properties and the beauty of all that is going on within one simple problem and its solution.

You will be using the rules for adding and multiplying positive and negative numbers from Hour 3 along with the seven properties you learned in this hour. Note that the answers can be written in different order—thanks to the commutative property—and either is correct.

Here are some examples with their solutions.

a. $2 + 5x + 9 + 3x = 11 + 8x$ or $8x + 11$
b. $-3x + 4 - 2x + (-6) = -5x - 2$ or $-5x + (-2)$
c. $(6 - 8) + 7x + 3(x - 4) = -2 + 7x + 3x - 12 = -14 + 10x$ or $10x - 14$
d. $7(n - 2) + 3(5 - n) = 7n - 14 + 15 - 3n = 4n + 1$ or $1 + 4n$
e. $5(3x - 2) - 3(4x + 3) = 15x - 10 - 12x - 9 = 3x - 19$ or $-19 + 3x$

In the previous example e, be sure that the -3 is distributed to the $4x$ and to the $+3$, resulting in $-12x$ and -9. In these more complex problems, it is important that you watch all the signs of the numbers very carefully, following the sign rules you learned in Hour 3.

Also, in examples c through e, the problem is followed by a middle step that leads to the final answer. You should always write this step as a part

of the solution because there are too many terms to keep track of as you work toward the answer.

Now try these problems and check your answers on page 42.

Simplify the following expressions:

1. $6a + 2b - 3a + 4b - c =$
2. $-3(x + 5) + 12 + 2(x - 4) =$
3. $-5 - 2(g + 3) + 10 =$
4. $8(2x - 3) - 20 + 2(5x + 6) =$
5. $x - 4 + 3(2x - 3) + 8 =$

Using Multiplication Rules with Real Numbers

Most of the examples and problems you have solved in Hours 1 through 4 have included whole numbers (0, 1, 2, 3 …) and their opposites (… –3, –2, –1), which, when combined, form the set of integers. You will need to be able to work with fractions in many algebra problems, and the reciprocal property of multiplication helps you work more quickly with fractions.

Consider this multiplication problem, which includes an integer (3), a variable (x), and a fraction $\left(\frac{1}{3}\right)$:

$$3x \cdot \frac{1}{3}$$

Fractions can cause a panic attack for many people, but using the reciprocal property will help you solve this and many fraction problems easily.

FYI A positive number always has a positive reciprocal. A negative number always has a negative reciprocal.

This property also applies to variables, so the reciprocal of a is $\frac{1}{a}$, and the reciprocal of x is $\frac{1}{x}$.

In Hour 5, when you learn how to solve equations, this reciprocal property will be an important part of the process.

For now, learn how to simplify expressions that include reciprocals, such as the following problem:

$$3x \cdot \frac{1}{3}$$

Because 3 and $\frac{1}{3}$ are reciprocals, multiplying them must equal 1. The expression is now $1 \cdot x$. Multiplying any number or term by 1 doesn't change the number or term (the identity property of multiplication). So the final answer is x. When you see a multiplication problem that includes reciprocals, simplifying the problem is easy.

Here are some examples of the reciprocal and identity properties of multiplication.

f. $\frac{1}{6} \cdot 6x = 1 \cdot x = x$

g. $8x \cdot \frac{1}{8} = 1 \cdot x = x$

h. $10x \cdot \frac{1}{5} = 5 \cdot 2 \cdot x \cdot \frac{1}{5} = 1 \cdot 2x = 2x$

i. $15xy \cdot \frac{1}{3} = 5 \cdot 3 \cdot xy \cdot \frac{1}{3} = 1 \cdot 5xy = 5xy$

j. $\frac{1}{3}(6x + 18) = \frac{1}{3} \cdot 6x + \frac{1}{3} \cdot 18 = \frac{1}{3} \cdot 3 \cdot 2x + \frac{1}{3} \cdot 3 \cdot 6 = 1 \cdot 2x + 1 \cdot 6 = 2x + 6$

k. $(-24ad)\left(-\frac{1}{8}\right) = -8 \cdot 3ad \cdot -\frac{1}{8} = 1 \cdot 3ad = 3ad$

Seeing the steps followed in examples f through k above, you will probably realize that some shortcuts are definitely possible. In some of these examples, you could have gone directly from the problem to the final answer, but writing out the steps that lead to the solution may help you avoid careless errors. Decide how much work you need to show and how much can be done mentally, thus creating the shortcuts.

Now try these problems and check your answers on pages 42–43.

6. $\frac{1}{3} \cdot 3x =$

7. $12y \cdot \frac{1}{12} =$

8. $14x \cdot \frac{1}{7} =$

9. $24ab \cdot \frac{1}{8} =$

10. $\frac{1}{4}(6a + 24) =$

11. $(-10xy)\left(-\frac{1}{5}\right) =$

12. $-\frac{1}{8}(6ad) =$

Using the Division Rule with Real Numbers

In Hour 3, you saw the close link between addition and subtraction. There is also a very strong connection between multiplication and division, and it makes the jobs of dividing algebraic expressions and reducing fractions easier.

Consider this simple division problem:

$$12 \div 4 = 3$$

And this multiplication problem:

$$12 \cdot \frac{1}{4} = 3 \cdot 4 \cdot \frac{1}{4} = 3 \cdot 1 = 3$$

These two problems show the close link between division and multiplication. Stated most simply, dividing by 4 is exactly the same as multiplying by $\frac{1}{4}$. Remember, too, that the reciprocal of 4 is $\frac{1}{4}$. Putting these facts together creates the division rule for real numbers: to divide by a number, multiply by its reciprocal.

PROCEED WITH CAUTION

The division rule applies to all real numbers except 0, since 0 doesn't have a reciprocal.

Of course, some division problems are easily solved without having to change them to multiplication. A good example is the simple division problem $12 \div 4 = 3$. But solving other problems requires changing to multiplication.

Examples l through p, with explanations after each one, show how the division rule is used in more complex problems:

l. $10 \div 2 = 10 \cdot \frac{1}{2} = 5 \cdot 2 \cdot \frac{1}{2} = 5 \cdot 1 = 5$

This problem can be solved quickly because you know from basic arithmetic that 10 divided by 2 equals 5. But the steps shown in the previous example follow the properties you have learned in this hour, and you will see how helpful they can be in the next example:

m. $10 \div \frac{1}{2} = 10 \cdot 2 = 20$

Dividing by $\frac{1}{2}$ is not a basic arithmetic skill, but the division rule makes it an easy problem.

n. $10x \div -\frac{1}{2} = 10x \cdot -2 = -20x$

JUST A MINUTE

Recall the sign rules for ÷. Dividing two positive numbers gives a positive answer. Dividing two negative numbers gives a positive answer. Dividing two numbers with different signs gives a negative answer.

By including a variable and a negative number, you are now doing an algebra problem, but the division rule still applies, and you know the rules for multiplying positive and negative numbers.

o. $\frac{10x}{2} = 10x \div 2 = 5 \cdot 2 \cdot x \cdot \frac{1}{2} = 5x$

This problem is written in fraction form. The solution steps show it then changing into a division problem. This is a good example of an algebra problem that you should be able to solve without showing any of the steps. Since $\frac{10}{2} = 5$, you easily get the correct answer of $5x$:

p. $-10 \cdot \left(\frac{x}{-2}\right) = -10 \cdot \left(x \div -2\right) = -10 \cdot \left(x \cdot -\frac{1}{2}\right) = 5 \cdot -2 \cdot \left(x \cdot -\frac{1}{2}\right) = 5x$

Here is a perfect example of an easily solved algebra problem, going from the problem to the final answer in one step. The steps shown here give all the small details, but in reality, you would simply go from $-10 \cdot \left(\frac{x}{-2}\right)$ to $5x$.

Now try these problems and check your answers on page 43.

13. $8 \div \frac{1}{2} =$
14. $14x \div 2 =$
15. $20a \div 10 =$

16. $\frac{-16x}{8} =$
17. $12d \div -3 =$
18. $-15 \cdot \left(\frac{x}{3}\right) =$
19. $6 \cdot \left(\frac{ab}{-2}\right) =$
20. $-18 \cdot \left(\frac{ab}{3}\right) =$

Answers to Sample Problems in Hour 4

1. $6a$ and $3a$ are like terms and, when subtracted, equal $3a$; $2b$ and $4b$ are like terms and, when added, equal $6b$; the c has no like term. Final answer: $3a + 6b - c$
2. Distributing the -3 gives $-3x + (-15)$; distributing the 2 gives $2x + (-8)$. Arrange the like terms together: $-3x + 2x$ and $-15 + (-8) + 12$. Combining like terms gives the final answer: $-x + (-11)$ or $-x - 11$
3. Distributing the -2 gives $-2g + (-6)$. Arrange the like terms together: $-5 + (-6) + 10$. Combining like terms gives the final answer: $-2g + (-1)$ or $-2g - 1$
4. Distributing the 8 gives $16x - 24$; distributing the 2 gives $10x + 12$. Arrange the like terms together: $16x + 10x$ and $-24 - 20 + 12$. Combining like terms gives the final answer: $26x + (-32)$ or $26x - 32$
5. Distributing the 3 gives $6x + (-9)$. Arrange the like terms together: $x + 6x$ and $-4 + (-9) + 8$. Combining like terms gives the final answer: $7x + (-5)$ or $7x - 5$
6. $\frac{1}{3}$ and 3 are reciprocals, so their product is 1. $1 \cdot x = x$
7. 12 and $\frac{1}{12}$ are reciprocals, so their product is 1. $1 \cdot y = y$
8. Rewriting 14 as $2 \cdot 7 \rightarrow 2 \cdot 7 \cdot x \cdot \frac{1}{7} = 2 \cdot 1 \cdot x = 2x$
9. Rewriting 24 as $3 \cdot 8 \rightarrow 3 \cdot 8 \cdot ab \cdot \frac{1}{8} = 3ab$
10. Distributing the $\frac{1}{4}$ gives
    $\frac{1}{4} \cdot 16a + \frac{1}{4} \cdot 24 = \frac{1}{4} \cdot 4 \cdot 4 \cdot a + \frac{1}{4} \cdot 4 \cdot 6 = 4a + 6$
11. Rewriting -10 as $-5 \cdot 2 \rightarrow -5 \cdot 2xy \cdot -\frac{1}{5} = 2xy$

12. Rewriting 16 as $8 \cdot 2 \rightarrow \; -\frac{1}{8} \cdot 8 \cdot 2cd = -2cd$
13. $8 \div \frac{1}{2} = 8 \cdot 2 = 16$
14. $14x \div 2 = 14x \cdot \frac{1}{2} = 7 \cdot 2x \cdot \frac{1}{2} = 7x$
15. $20a \div 10 = 20a \cdot \frac{1}{10} = 2 \cdot 10a \cdot \frac{1}{10} = 2a$
16. $\frac{-16x}{8} = -16x \div 8 = -16x \cdot \frac{1}{8} = 8 \cdot -2x \cdot \frac{1}{8} = \cancel{2}x$
17. $12d \div -3 = 12d \cdot -\frac{1}{3} = 3 \cdot 4d \cdot -\frac{1}{3} = -4d$
18. $-15 \cdot \left(\frac{x}{3}\right) = -15 \cdot \left(x \div 3\right) = -15 \cdot \left(x \cdot \frac{1}{3}\right) = 3 \cdot -5\left(x \cdot \frac{1}{3}\right) = -5x$
19. $6 \cdot \left(\frac{ab}{-2}\right) = 6 \cdot \left(ab \div -2\right) = 6 \cdot \left(ab \cdot -\frac{1}{2}\right) = 2 \cdot 3\left(ab \cdot -\frac{1}{2}\right) = -3ab$
20. $-18 \cdot \left(\frac{ab}{3}\right) = -18 \cdot \left(ab \div 3\right) = -18 \cdot \left(ab \cdot \frac{1}{3}\right) = 3 \cdot -6 \cdot \left(ab \cdot \frac{1}{3}\right) = -6ab$

Hour's Up!

Review

Based on what you've learned in Hour 4, simplify the following algebraic expressions into simplest terms. Check your answers with the solution key in Appendix B.

1. $5x + 3y - 2x - 4y + 3z =$
- **a.** $3x - 7y + 3z$
- **b.** $-7x - 7y + 3z$
- **c.** $3x - y + 3z$
- **d.** $3x - y + z$

2. $-6(x + 2) + 10 + 3(x - 3) =$
- **a.** $3x - 11$
- **b.** $-3x - 11$
- **c.** $3x + 11$
- **d.** $-3x + 11$

3. $-4(a + 5) - 3(a - 4) + 8 =$
- **a.** $-7a$
- **b.** $-7a + 28$
- **c.** $-a + 28$
- **d.** $-a + 20$

4. $\frac{1}{4} \cdot 4d$

a. d
b. $4d$
c. 4
d. $16d$

5. $16a \cdot \frac{1}{2} =$

a. $14a$
b. $32a$
c. 32
d. $8a$

6. $\frac{1}{6}(12x - 18) =$

a. $2x + 3$
b. $3x - 2$
c. $2x - 3$
d. $-x$

7. $-\frac{1}{2}(-18xy) =$

a. $9xy$
b. $-9xy$
c. $36xy$
d. $-36xy$

8. $9 \div -\frac{1}{3} =$

a. 3
b. -3
c. -27
d. 27

9. $-20y \div 4 =$

a. $-5y$
b. $5y$
c. -5
d. $-80y$

10. $-24 \cdot \left(-\frac{xy}{4}\right)$

a. $20xy$
b. $-6xy$
c. $6xy$
d. $-20xy$

Review

Part II

Solving Equations

Hour 5 Linear Equations: Solving with Basic Operations

Hour 6 Linear Equations: Solving More Complex Equations

Hour 5

Linear Equations: Solving with Basic Operations

CHAPTER SUMMARY

In this hour you will learn about …

- Solving equations with addition and subtraction
- Solving equations with multiplication and division
- Solving practical problems with equations

If one process sets algebra apart from basic arithmetic, it is solving equations. During your first four hours of study, you learned about many properties and rules that serve as the foundation of algebraic knowledge. This hour, you will learn about four more properties that, when applied to the others, give you the tools to solve equations.

Solving Equations with Addition and Subtraction

You know that an equation is simply two expressions that are equal to each other, and the equality is shown with an equals sign. Basic equations are like $2 + 3 = 5$ or $-6 + 9 = 3$. These equations are basic because they don't include any variables. If you add 2 and 3, the answer is 5. If you add –6 and 9, the answer is 3.

Equations can be true, false, or open. An equation is true if the expression on the left side of the equals sign is equal to the expression on the right side, as in $2 + 3 = 5$. An example of a false equation is $2 + 4 = 5$ because if you add 2 and 4, the answer is not 5; it's 6. Equations that include variables, such as $x + 8 = 10$ or $a - 5 = 1$, are called *open* because they can be true or false, depending on what number you substitute for the variable. Your job in solving an open equation is to find the number that, when substituted for the variable, makes the equation true.

STRICTLY DEFINED

Open equations have a variable. Replacing the variable with a number will make the equation either true or false. Solving an equation means finding the value of the variable that will make the equation true.

To say it another way, an equation is solved when the variable's numerical value is found. In the previous examples, $x + 8 = 10$ and $a - 5 = 1$, the solutions are $x = 2$ and $a = 6$. These equations are simple ones, and you probably knew the answers when you first saw the equations, using mental math.

But the actual process of solving equations involves following one of the four properties of equality, which you will learn through the next few examples, the first involving addition.

Begin by looking at a true equation from basic arithmetic:

$$4 + 3 = 7$$

Would the equation still be true if you add 2 to the left side and also add 2 to the right side? Let's see:

$$(4 + 3) + 2 = 7 + 2$$

You get 9 on the left and 9 on the right, so this is still a true equation. This is an example of the *addition property of equality*.

STRICTLY DEFINED

The **addition property of equality** lets you add the same amount to both sides of a true equation to create a new equation that is also true.

More examples from basic arithmetic follow:

In basic arithmetic, adding the same number to equal numbers gives equal sums.

Examples: $1 + 5 = 6$ is true, so $(1 + 5) + 4 = 6 + 4$; and $-3 + 5 = 2$ is true, so $(-3 + 5) + 1 = 2 + 1$

In algebra, adding the same amount to equal expressions gives equal sums.

Examples: If $x = y$, then $x + z = y + z$; and if $b = h$, then $b + d = h + d$.

The *subtraction property of equality* is similar. Begin again with a true equation from basic arithmetic:

$$2 + 7 = 9$$

Would the equation still be true if you subtract 5 from the left side and also subtract 5 from the right side? Let's see:

$$(2 + 7) - 5 = 9 - 5$$

Yes, both sides of this new equation equal 4, so the new equation is true. This is an example of the *subtraction property of equality.*

STRICTLY DEFINED

The **subtraction property of equality** lets you subtract the same amount from both sides of a true equation to create a new equation that is also true.

In basic arithmetic, subtracting the same number from equal expressions gives equal differences.

Examples: $8 + 2 = 10$ is true, so $(8 + 2) - 4 = 10 - 4$; and $-7 + 4 = -3$ is true, so $(-7 + 4) - 5 = -3 - 5$.

In algebra, subtracting the same amount from equal expressions gives equal differences.

Examples: If $x = y$, then $x - z = y - z$; and if $b = b$, then $b - d = b - d$.

When you look at a given equation, you can see that it tells you a lot of information, but not exactly what you want to know. You want to know the solution, the value for the variable that makes the equation true. For example, the equation $x + 5 = 12$ is saying that x added to 5 is the same as 12. But you want to know what x equals, not what $x + 5$ equals.

Your goal in solving the equation is to get the x alone on the left side of the equals sign and a number (the solution) on the right side. The 5 that is added to the x is actually in your way and is keeping you from knowing what x equals. It would be good if you could make the left side of the equation change from $x + 5$ to x, and then x would be alone on the left side.

What would it take to change the 5 into a 0? Subtracting 5 from 5 gives 0. The subtraction property allows you to subtract 5 from the left side as long as you also subtract it from the right side of the equation. Changing

the 5 into a 0 will be very helpful because instead of having $x + 5$ on the left side of the equation, you will have $x + 0$, which is the same as x. (See the sidebar.)

STRICTLY DEFINED

The **identity property of addition** states that if you add 0 to any number or term, the answer is the number or term itself. $4 + 0 = 4$; $b + 0 = b$

Let's go back to the beginning of the sample equation and solve it, showing the steps that lead to the solution:

Solve: $x + 5 = 12$
$x + 5 - 5 = 12 - 5$
$x + 0 = 7$
$x = 7$

You should always check your solution by substituting the solution into the original equation and verifying that it works. Yes, $7 + 5 = 12$, and your solution $x = 7$ is correct. You probably knew that $x = 7$ is the solution because the original equation was an easy one to solve mentally. But the steps that include the subtraction property of equality and the identity property of addition show the algebraic process at work.

This process is the same for simple and more complex equations that are not so easily done with mental arithmetic. Learn how to perform the steps by working on easier problems first, and then the tougher equations will be easier to solve.

See how the subtraction property of equality is applied in the next example:

Solve: $a + 3 = -7$
$a + 3 - 3 = -7 - 3$
$a = -10$

To check the solution, put -10 in for a in the original equation: $-10 + 3 = -7$; therefore, $a = -10$ is correct.

Notice that in the solution steps of the previous equation, the step $a + 0 = -10$ isn't shown, as $x + 0 = 7$ was in the previous example. You are free to omit this step in your work as an acceptable shortcut. If you are more comfortable showing the step, that's fine, too.

STRICTLY DEFINED

Remember that **solving an equation** means finding the value of the variable that makes the equation true.

The last thing you need to know about solving these types of equations is what to do when the variable is on the right side. Your goal remains to isolate the variable, no matter where it is in the equation. The next example has the variable on the right side:

Solve: $3 = x + 5$
$3 - 5 = x + 5 - 5$
$-2 = x$ or $x = -2$

To check, put –2 in the original equation for x: $3 = -2 + 5$ is true. Therefore, $x = -2$ is the solution.

Now try these problems and check your answers on page 56.

Solve:

1. $x + 4 = -5$
2. $b - 5 = -6$
3. $2 = x - 9$

The equations you have solved so far involve adding and subtracting. Let's consider now how to work with equations that have multiplication or division of a variable.

Solving Equations with Multiplication or Division

Your goal is still to isolate the variable, and you'll be using one of the two remaining properties of equality, which you will learn through the next few examples.

Begin by looking at a true equation from basic arithmetic:

$3 + 2 = 1 + 4$

It's true because $5 = 5$.

What will happen if you multiply both the left and right sides by 6? Is it true that $6(3 + 2) = 6(1 + 4)$? Yes, $6 \cdot 5 = 6 \cdot 5$. This is an example of the *multiplication property of equality*, and more examples from basic arithmetic and algebra follow:

STRICTLY DEFINED

The **multiplication property of equality** lets you multiply the same amount to both sides of a true equation to create a new equation that is also true.

In basic arithmetic, multiplying equal numbers by the same number gives equal products.

Examples: $1 + 5 = 6$ is true, so $(1 + 5) \cdot 4 = 6 \cdot 4$; $-3 + 5 = 2$ is true, so $3 \cdot (-3 + 5) = 3 \cdot 2$

In algebra, multiplying equal expressions by the same amount gives equal products.

Examples: If $x = y$, then $x \cdot z = y \cdot z$; and if $b = h$, then $b \cdot d = h \cdot d$.

As you might expect, the *division property of equality* is similar.

Begin again with a true equation from basic arithmetic:

$$2 + 4 = 6$$

Would the equation still be true if you divide both sides by 3? Let's see:

$$(2 + 4) \div 3 = 6 \div 3$$

Yes, both sides of this equation equal 2, and the new equation is true. This is an example of the *division property of equality*, and more examples follow.

STRICTLY DEFINED

The **division property of equality** lets you divide both sides of a true equation by the same number to create a new equation that is also true.

In basic arithmetic, dividing equal numbers by the same number gives equal quotients.

Examples: $1 + 5 = 6$ is true, so $(1 + 5) \div 2 = 6 \div 2$; $-3 + 13 = 10$ is true, so $(-3 + 13) \div 2 = 10 \div 2$.

In algebra, dividing equal expressions by the same number gives equal quotients.

Examples: If $x = y$, then $x \div z = y \div z$; if $b = h$, then $b \div d = h \div d$.

Now that you know about the multiplication and division properties of equality, you are ready to use them to solve equations. The process is just like solving equations using addition and subtraction.

Consider the following equation. What will it take to find its solution?

Solve: $3x = 15$

The equation is telling you that 3 times x is equal to 15. To solve it, you want to know what value of x makes the equation true. You want the x to be isolated on the left side of the equation. The 3 that is multiplied times the x is in the way and is keeping you from knowing what x equals. It would be good if you could make the left side of the equation change from $3 \cdot x$ to $1 \cdot x$ because $1 \cdot x = x$ (identity property of multiplication), and then x would be alone on the left side.

What would it take to change the 3 into a 1? Dividing 3 by 3 gives 1. The division property allows you to divide by 3 on the left side as long as you also divide it on the right side of the equation.

PROCEED WITH CAUTION

Be careful: division is not commutative. The order of the numbers in division is important: $12 \div 4$ does not equal $4 \div 12$.

Let's go back to the beginning of the sample equation and solve it, showing the steps that lead to the solution:

Solve: $3x = 15$
$3x \div 3 = 15 \div 3$
$1x = 5$
$x = 5$

To check, put 5 in for x in the original equation: $3 \cdot 5 = 15$ is true. Therefore, $x = 5$ is the solution. This equation can easily be solved mentally, but just as with the addition and subtraction equations, learning how to use the properties correctly on the easier equations will help you be able to solve the more difficult ones, where you will need to show some or all of the steps.

Now use the multiplication property to solve this sample equation:

Solve: $x \div 8 = 2$
$x \div 8 \cdot 8 = 2 \cdot 8$
$x = 16$

To check, put 16 in for x in the original equation: $16 \div 8 = 2$ is true. Therefore, $x = 16$ is the solution.

The next example has the variable on the right side, but the process is the same:

Solve: $24 = -6x$
$24 \div -6 = -6x \div -6$
$-4 = x$ or $x = -4$

To check, put -4 in for x in the original equation: $24 = -6 \cdot -4$ is true. Therefore, $x = -4$ is the solution.

Now try these problems and check your answers on page 56.

Solve:

4. $-5x = 15$
5. $a \div 7 = 3$
6. $c \div 4 = -8$
7. $-10 = -2x$

Solving Practical Problems with Linear Equations

Having the skills to solve equations makes it possible for you to also solve real-world problems, often called word problems. Here's one example: David ran the 400-meter dash in 62.6 seconds, which is 2.5 seconds faster than his previous best time. What was his previous best time?

Write and solve an equation, using b for David's previous best time:

$62.6 = b - 2.5$
$62.6 + 2.5 = b - 2.5 + 2.5$
$65.1 = b + 0$
$65.1 = b$

GO TO ◄

equations for real-life problems are in Hour 1.

To check, put 65.1 in for b in the original equation: $62.6 = 65.1 - 2.5$ is true. Therefore, $b = 65.1$ is the solution. David's previous best time was 65.1 seconds.

If a situation can be written as an equation with one of the four operations, you can find the answer.

Here are some more examples that show how to answer real-life questions using equations:

> The home team won 30 games this season, which is 5 times as many games as they lost. How many games did they lose? Write and solve an equation, using x for the number of games they lost.
> $30 = 5 \cdot x$
> $30 \div 5 = 5 \cdot x \div 5$
> $6 = x$ or $x = 6$

To check, put 6 in for x in the original equation: $30 = 5 \cdot 6$ is true. Therefore, $x = 6$ is the solution. The team lost 6 games.

If each item in a box weighs 2 pounds, how many items are packed in a box that holds 70 pounds?

Write and solve an equation, using x for the number of items.

> $2x = 70$
> $2x \div 2 = 70 \div 2$
> $x = 35$

To check, put 35 in for x in the original equation: $2 \cdot 35 = 70$ is true. Therefore, $x = 35$ is the solution. 35 items are packed in the box.

In algebra, there will be times when you have to solve equations that are written in word form instead of algebraic form. Here is an example: If a number is increased by 14, the result is 20. Find the number.

Solution: Write and solve an equation, using n for the number:

> $n + 14 = 20$
> $n + 14 - 14 = 20 - 14$
> $n = 6$

To check, put 6 in for n in the original equation: $6 + 14 = 20$ is true. Therefore, $x = 6$ is the solution. 6 is the number you are looking for in this problem.

Another example: Thirty less than a number is negative eight. Find the number.

Solution: Write and solve an equation, using n as the number:

$$n - 30 = -8$$
$$n - 30 + 30 = -8 + 30$$
$$n = 22$$

To check, put 22 in for n in the original equation: $22 - 30 = -8$ is true. Therefore, $n = 22$ is the solution.

Now try these problems and check your answers below. Write an equation for each problem, and use the four properties of equality to solve the equation.

8. The temperature rose 25°F from the low temperature at 6:30 A.M. to a record high of 97°F, which occurred at 2:30 P.M. What was the low temperature at 6:30 A.M.?
9. Your online bank record shows a balance of $342.50 today. Your own checkbook shows a balance of $467.80, and all the deposits and withdrawals in your checkbook match those in the bank's online statement. What amount of money will you need to add into your account so it will not be overdrawn?
10. How many pieces of candy that have 20 calories each can you eat if your diet allows you to have a snack that totals 120 calories?
11. A number decreased by 12 is 9. Find the number.
12. A number increased by 5 is –6. Find the number.

Answers to Sample Problems in Hour 5

1. $x + 4 = -5 \rightarrow x + 4 - 4 = -5 - 4 \rightarrow x + 0 = -9 \rightarrow x = -9$
2. $b - 5 = -6 \rightarrow b - 5 + 5 = -6 + 5 \rightarrow b + 0 = -1 \rightarrow b = -1$
3. $2 = x - 9 \rightarrow 2 + 9 = x - 9 + 9 \rightarrow 11 = x$ or $x = 11$
4. $-5x = 15 \rightarrow -5x \div -5 = 15 \div -5 \rightarrow 1 \cdot x = -3 \rightarrow x = -3$
5. $a \div 7 = 3 \rightarrow a \div 7 \cdot 7 = 3 \cdot 7 \rightarrow a \cdot 1 = 21 \rightarrow a = 21$
6. $c \div 4 = -8 \rightarrow c \div 4 \cdot 4 = -8 \cdot 4 \rightarrow c \cdot 1 = -32 \rightarrow c = -32$
7. $-10 = -2x \rightarrow -10 \div -2 = -2x \div -2 \rightarrow 5 = 1 \cdot x \rightarrow 5 = x$ or $x = 5$
8. $x + 25 = 97 \rightarrow x + 25 - 25 = 97 - 25 \rightarrow x + 0 = 72 \rightarrow x = 72$ (72°F)
9. $342.50 + x = 467.80 \rightarrow 342.50 + x - 342.50 = 467.80 - 342.50 \rightarrow x = 125.30$. You need to deposit $125.30 to not be overdrawn.
10. $20x = 120 \rightarrow 20x \div 20 = 120 \div 20 \rightarrow x = 6$. You can have 6 pieces.
11. $x - 12 = 9 \rightarrow x - 12 + 12 = 9 + 12 \rightarrow x + 0 = 21 \rightarrow x = 21$
12. $x + 5 = -6 \rightarrow x + 5 - 5 = -6 - 5 \rightarrow x + 0 = -11 \rightarrow x = -11$

Hour's Up!

Review

Based on what you've learned in Hour 5, solve the following equations. Check your answers with the solution key in Appendix B.

Solve the following equations:

1. $x + 6 = 14$
- **a.** $x = -8$
- **b.** $x = 20$
- **c.** $x = -20$
- **d.** $x = 8$

2. $c - 10 = -3$
- **a.** $c = -7$
- **b.** $c = 7$
- **c.** $c = 13$
- **d.** $c = -13$

3. $6 = a - 5$
- **a.** $a = 11$
- **b.** $a = -11$
- **c.** $a = -1$
- **d.** $a = 1$

4. $-3 = x + 8$
- **a.** $x = -5$
- **b.** $x = 5$
- **c.** $x = 11$
- **d.** $x = -11$

5. $3x = -18$
- **a.** $x = -6$
- **b.** $x = 15$
- **c.** $x = -15$
- **d.** $x = 6$

6. $b \div 5 = -4$
- **a.** $b = -9$
- **b.** $b = -20$
- **c.** $b = -1$
- **d.** $b = 20$

Review

7. $-12 = -6x$
a. $x = 6$
b. $x = -6$
c. $x = 2$
d. $x = -2$

8. Your bank balance this morning was $210.40. You made a deposit today, resulting in a new balance of $389.68, but you can't find the deposit slip that shows the amount of deposit. There were no other deposits or withdrawals in your account. How much did you deposit?
a. $79.28
b. $210.40
c. $129.28
d. $179.28

9. Cloud cover followed by a snowfall caused the temperature to fall 6°F from the day's high temperature, so now it is –2°F. What was the day's high?
a. –8°F
b. 8°F
c. –4°F
d. 4°F

10. 38 less than a number is –4. What is the number?
a. 34
b. 42
c. –34
d. –42

Hour 6

Linear Equations: Solving More Complex Equations

CHAPTER SUMMARY

In this hour you will learn about …

- Solving linear equations with more than one operation
- Solving linear equations with variables on both sides of the equals sign
- Solving linear equations with many or no solutions
- Solving practical problems with linear equations

Now that you can solve simple equations that have just one operation, the next goal is to learn how to solve more complex equations. These new equations contain two or more of the basic operations (addition, subtraction, multiplication, and division). You solve them using more than one of the previous hour's four properties of equality in a logical order. The challenge comes in knowing which property to use and when to use it. Before the end of this hour, you will be able to rearrange any linear equation's terms to isolate the variable—in other words, to solve the equation.

Solving Equations with Combined Operations

How can you decide which is the better deal for a two-day trip if one car-rental company charges $33 per day with unlimited miles and another charges $20 per day and 10¢ per mile? The answer depends on how many miles you plan to drive, and knowing the break-even point will help you make your decision. This real-life problem is an excellent example of using algebra to find a useful answer, and you will be able to solve this and similar problems at the end of this hour.

Take a moment to review an equation from Hour 5:

Solve $x - 9 = -1$.

To isolate the variable, add 9 to both sides of the equation and then simplify the expressions:

$$x - 9 + 9 = -1 + 9$$
$$x + 0 = 8$$
$$x = 8$$

To check, put 8 in for x in the original equation: $8 - 9 = -1$ is true. Therefore, $x = 8$ is the solution.

In the previous example, only one operation (subtraction) is involved, and you simply apply the addition property of equality to isolate the variable. Now consider a more complex equation that has two operations:

$$2x + 3 = 9$$

GO TO ◄
The properties of equality for +, –, ·, and ÷ help you move terms from one side of an equation to the other when used in the right order. See Hour 4 for a full set of examples of these properties.

Isolating the variable requires moving the 2 and also moving the 3. Which should you do first? Let's try moving the 3 by subtracting 3 from both sides of the equation:

$$2x + 3 = 9$$
$$2x + 3 - 3 = 9 - 3$$
$$2x + 0 = 6$$
$$2x = 6$$

Can you now move the 2? Yes, the division property of equality lets you divide both sides by 2. To finish solving the equation:

$$2x = 6$$
$$2x \div 2 = 6 \div 2$$
$$x = 3$$

To check, put 3 in for x in the original equation: $2 \cdot 3 + 3 = 9 \rightarrow 6 + 3 = 9 \rightarrow 9 = 9$, which is true. Therefore, $x = 3$ is the solution. Look once again at the original equation:

$$2x + 3 = 9$$

Instead of subtracting 3 from both sides of the equation, what would have happened if you had tried to move the 2 first? Moving the 2 requires dividing both sides of the equation by 2. But isn't the 3 in your way? So dividing both sides by 2 is not going to work here. By moving

the 3 first, you get an equation $2x = 6$ that can easily be solved when you divide both sides by 2.

Solving $2x + 3 = 9$ is a perfect example of the power of the properties of algebra. Being able to solve this type of equation builds up your math skills, taking you beyond the simple arithmetic operations. You are faced with choosing between two equally useful properties (subtraction and division) and then applying them in the correct order to find the solution.

PROCEED WITH CAUTION

Many of the simpler equations in Hour 5 could easily be solved mentally. $2x + 3 = 9$ has just enough complexity that you don't see the solution right away. Yet by applying two properties of equality, you can solve it quite easily. Showing the solution steps helps you avoid careless errors.

This higher-level algebra process gives you a new perspective that allows you to solve problems that can't be solved using basic arithmetic. At the end of this hour, you will see real-life situations that can be written as algebraic equations and solved. The solutions to these more complex equations are used extensively in research, finance, business, and medicine. Before you learn how to write equations for various situations, learn from the following examples how to solve equations that contain more than one operation.

Consider the following equation:

$$\frac{x - 8}{2} = 5$$

It has two operations (subtraction and division), so its solution needs at least two steps. Should you first add 8 to both sides or multiply both sides by 2?

Adding 8 to the fraction $\frac{x - 8}{2}$ will not make the –8 become a 0, so try multiplying both sides by 2. The steps look like this:

$$\frac{x - 8}{2} \cdot 2 = 5 \cdot 2$$

$$x - 8 = 10$$

Now adding 8 to both sides gives the solution $x = 18$.

GO TO ◄
Use the Sign Rules from Hour 3 for adding positive and negative numbers when combining like terms.

To check, put 18 in for x in the original equation:

$\frac{18-8}{2} = 5 \rightarrow \frac{10}{2} = 5 \rightarrow 5 = 5$. Therefore, $x = 18$ is the solution.

Some equations have like terms that need to be combined before they can be solved, as in this sample equation: $3x + 2x - 8 = 3 + 9$

The $3x$ and $2x$ are like terms, and 3 and 9 are also like terms. Combine them, and the equation becomes this:

$$5x - 8 = 12$$

Solve it using the addition and division properties of equality, deciding which to do first. Dividing both sides by 5 won't work because the left side is $5x - 8$. You need to first move the 8 by adding 8 to both sides. The steps are as follows:

$5x - 8 = 12$
$5x - 8 + 8 = 12 + 8$
$5x = 20$
$5x \div 5 = 20 \div 5$
$x = 4$

To check, put 4 in for x, being careful to use the original equation:

$3 \cdot 4 + 2 \cdot 4 - 8 = 3 + 9 \rightarrow 12 + 8 - 8 = 12 \rightarrow 20 - 8 = 12 \rightarrow 12 = 12.$
Therefore, $x = 4$ is the solution.

The following example includes combining like terms and working with negative numbers, but you still use the four properties of equality to get the solution:

Solve $-4a + 3 - 2a + 10 = -5$.

Combine the like terms and use the division and subtraction properties to solve it:

$-4a + 3 - 2a + 10 = -5$
$-6a + 13 = -5$
$-6a + 13 - 13 = -5 - 13$
$-6a = -18$
$-6a \div -6 = -18 \div -6$
$a = 3$

To check, put 3 in for a in the original equation: $-4 \cdot 3 + 3 - 2 \cdot 3 + 10 = -5 \rightarrow -12 + 3 - 6 + 10 = -5 \rightarrow -9 + -6 + 10 = -5 \rightarrow -15 + 10 = -5 \rightarrow -5 = -5$. Therefore, $a = 3$ is the solution.

Now try solving these equations, checking your answers on page 69.

1. $5x - 4 = 11$
2. $-3a - 6 = 9$
3. $10 + 4c - 5 + 2c = -7$
4. $\frac{x - 4}{3} + 6 = -2$

The car rental problem can be solved using an equation like those you just did, but it is better to wait and do all the word problems at the end of the hour—there are more types of equations to learn about first.

Solving Equations with the Variable on Both Sides

One company offers you an annual salary of $18,000 plus a $1,500 raise each year, and another offers an annual salary of $20,000 and a $1,000 raise each year. Assuming both jobs are equally appealing to you in all other ways, which of these two pay scales should you choose? The answer depends on how many years you work there. This problem is written as an equation with a variable on both sides of the equals sign, and it has a slightly different solution process. You will see this process shown in some sample equations as you work toward solving it and other word problems the end of the hour.

Follow the steps to solve for x in the next example:

$$7x = 4x + 15$$

Your goal, as always, is to isolate the variable. With an x on both sides of the equals sign, which variable will you isolate? The answer is that isolating the variable has to wait because the two x terms must first be combined. The equals sign separates the $7x$ from the $4x$, so your first job is moving either the $7x$ or the $4x$ to the other side of the equals sign.

JUST A MINUTE

Remember that the properties of equality allow you to move numbers from one side to the other until you have isolated the variable. These same properties make it possible to move terms with variables, too.

Look again at the equation you are trying to solve:

$$7x = 4x + 15$$

The subtraction property of equality lets you subtract either $7x$ or $4x$ from both sides of the equation. Try subtracting $4x$ from both sides and see the result:

$$7x - 4x = 4x + 15 - 4x$$
$$3x = 15$$

Dividing both sides of the equation by 3 isolates the variable, and you have the solution:

$$3x \div 3 = 15 \div 3$$
$$x = 5$$

To check, put 5 in for x in the original equation:

$$7 \cdot 5 = 4 \cdot 5 + 15 \rightarrow 35 = 20 + 15 \rightarrow 35 = 35$$

Therefore, $x = 5$ is the solution.

You did have two choices in solving this equation: you could have subtracted $7x$ or $4x$ from both sides in the first step. See what would have happened if you had subtracted $7x$ from both sides of the original equation:

$$7x - 7x = 4x + 15 - 7x$$
$$0 = -3x + 15$$
$$0 - 15 = -3x + 15 - 15$$
$$-15 = -3x$$
$$-15 \div -3 = -3x \div -3$$
$$5 = x \text{ or } x = 5$$

The solution is still $x = 5$, but it took many more steps to find the solution. In general, moving the variable term with the smaller *coefficient* requires fewer solution steps.

STRICTLY DEFINED

Remember that a **coefficient** is the number directly in front of a variable and is multiplied by the variable. For example, the coefficient of $7x$ is 7 and the coefficient of $-3a$ is -3.

Another type of equation includes parentheses that are affected by the distributive property. Some simplifying steps are needed before you can move any of the terms from one side to the other.

Follow these steps that show how to find the next equation's solution:

$$6(x - 2) + x = 2(x + 5) + 3$$
$$6x - 12 + x = 2x + 10 + 3$$
$$7x - 12 = 2x + 13$$
$$7x - 12 - 2x = 2x + 13 - 2x$$
$$5x - 12 = 13$$
$$5x - 12 + 12 = 13 + 12$$
$$5x = 25$$
$$5x \div 5 = 25 \div 5$$
$$x = 5$$

To check, put $x = 5$ in for x in the original equation:

$$6(5 - 2) + 5 = 2(5 + 5) + 3 \rightarrow 6 \cdot 3 + 5 = 2 \cdot 10 + 3 \rightarrow 18 + 5 = 20 + 3 \rightarrow 23 = 23.$$

Therefore, $x = 5$ is the solution.

JUST A MINUTE

From Hour 4: The Distributive Property

Multiplying a sum by a number is the same as adding the product of the numbers. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Now try solving these equations, checking your answers on pages 69–70.

5. $12x + 3 = 10x + 7$
6. $6 - 2x = 3x - 9$
7. $-6x + 8 = 2x$
8. $4(a - 5) = -24$

There's just one more detour before you learn how to write the equation and find the answer to the rental car and salary questions.

In all the equations you have solved so far, each solution was an integer and each equation had only one solution. To make it easier for you to learn the algebraic processes for solving equations, the problems have been chosen so that each solution is an integer. However, solutions can

be any real number, which you recall includes rational and irrational numbers. Fractions and decimal solutions are certainly possible and are common in real-life situations.

Solving Equations with No Solutions or Many Solutions

Take a moment to consider two questions. Does every equation have a solution? Can there be more than one solution to an equation?

See what happens as you follow the steps in the next equation:

$$8x = 2(4x + 3)$$
$$8x = 8x + 6$$
$$8x - 8x = 8x + 6 - 8x$$
$$0 = 6$$

It is not possible for 0 to equal 6, so something must have gone wrong in solving this equation. But as you carefully follow each of the steps in the solution, you see that each step is the result of an algebraic property, and everything is done correctly.

So how can there be a solution of $0 = 6$? The original equation looked perfectly fine, and the steps were going along in order, but the x terms disappeared as the process moved from the third to the fourth step. Whenever this occurs, the equation is said to have "no solution," and that is the answer that should be given for this equation. There is no numerical value that can be substituted in place of x to make this equation a true statement. The equation has no solution.

PROCEED WITH CAUTION

It is possible for an equation to have no solution or to have an infinite number of solutions—all real numbers!

Now follow the steps in another sample equation:

$$3x + 2 = 2(x - 5) + x + 12$$
$$3x + 2 = 2x - 10 + x + 12$$
$$3x + 2 = 3x + 2$$
$$3x + 2 - 3x = 3x + 2 - 3x$$
$$2 = 2$$

Solving the previous equation causes the x terms to disappear, just as they did in the earlier example. But this time the steps end with a true statement of 2 = 2. When this happens, the solution is "all the real numbers" because 2 does equal 2. No matter what number you use for x in the original equation, the steps will always lead you to a number equaling itself. The solution to this type of equation is all the real numbers.

Seldom will an equation's solution be "all the real numbers" or "no solution," but you need to be prepared when it happens.

Now try solving these equations, checking your answers on page 70.

9. $6b + 8 = 2(3b + 5)$
10. $6b - 24 = 3(2b - 8)$

Solving Practical Problems Using Linear Equations

Now it's time to put your equation-solving skills to work in real-life situations like the rental car and salary offer problems. As in Hour 1, the first task is to write the algebraic equation based on the given information.

GO TO ◄
Refer to Hour 1 for the table of word phrases and their algebra operations.

Let's begin with the rental car problem from earlier this hour. How can you decide which is the better deal for a two-day trip when one car-rental company charges $33 per day with unlimited miles and another charges $20 per day and 10¢ per mile? This question is easily answered by writing and solving an equation. First decide what the variable is in this situation. The number of days doesn't vary; it's 2. But the number of miles you will drive can change, so m for miles is the variable for the equation.

By finding out how many miles will make the two companies' costs the same (equal), you'll be able to choose the better deal based on the number of miles you plan to drive. As you write the equation, begin with the first company, which charges $33 for each of the two days. You want to know when that equals the other company's rate, which is $20 for each of the two days plus 10¢ per mile.

Here's the information in equation form, followed by the steps to find the solution. You may want to use a calculator for the larger numbers and decimals that often appear in real-life word problems.

$$33 \cdot 2 = 20 \cdot 2 + .10 \cdot m$$
$$66 = 40 + .10 \cdot m$$
$$66 - 40 = 40 + .10 \cdot m - 40$$
$$26 = .10 \cdot m$$
$$26 \div .10 = .10 \cdot m \div .10$$
$$260 = m$$

To check, put 260 into the original equation:

$$33 \cdot 2 = 20 \cdot 2 + .10 \cdot 260$$
$$66 = 40 + 26$$
$$66 = 66$$

Therefore, 260 miles is the break-even point, where the cost for your trip will be the same at both companies. Since the first company allows you to drive unlimited miles, it's the better deal if you plan to go over 260 miles. If your trip is less than 260 miles, the second deal will save you some money. Of course, taxes are going to be added to the cost, but at least you have the number of 260 miles to help make your final decision, all thanks to algebra!

Look again at the salary question. One company offers you an annual salary of $18,000 plus a $1,500 raise each year, and another company offers an annual salary of $20,000 and a $1,000 raise each year. Assuming both jobs are equally appealing to you in all other ways, which of these two pay scales should you choose? The answer depends on how many years you plan to work at the company. Knowing how many years it takes for the salaries to equal each other will help you make the decision. The number of years is the variable here, so use y for years.

The first company's offer is $18,000 plus $1,500 times y for each year that goes by. The second offer is $20,000 plus $1,000 times y for each year that passes. Write these as algebraic expressions and see when they are equal:

$$18{,}000 + 1{,}500 \cdot y = 20{,}000 + 1{,}000 \cdot y$$
$$18{,}000 + 1{,}500 \cdot y - 1{,}000 \cdot y = 20{,}000 + 1{,}000 \cdot y - 1{,}000 \cdot y$$
$$18{,}000 + 500 \cdot y = 20{,}000$$
$$18{,}000 + 500 \cdot y - 18{,}000 = 20{,}000 - 18{,}000$$
$$500 \cdot y = 2{,}000$$
$$500 \cdot y \div 500 = 2{,}000 \div 500$$
$$y = 4$$

To check, put 4 in for y in the original equation:

$$18{,}000 + 1{,}500 \cdot 4 = 20{,}000 + 1{,}000 \cdot 4$$
$$18{,}000 + 6{,}000 = 20{,}000 + 4{,}000$$
$$24{,}000 = 24{,}000$$

In four years, the $18,000 annual salary with $1,500 raises each year will equal the pay you will get at the other company. After four years, you will be earning more at the company that offered you only $18,000 to start. Setting up and solving the equation allows you to make the comparison and gives you the number of years it takes for the salaries to be the same. The final decision is still up to you.

Now try writing and solving these equations, checking your answers on page 70.

11. Sue has $125 in her daughter's piggy bank. How long will it take for her to have $200 if she adds $7.50 each week?
12. Find a number whose product with 8 is the same as its sum with 35.
13. Five times a number, increased by 12, is the same as twice the number increased by 36. Find the number.

Answers to Sample Problems in Hour 6

1. $5x - 4 = 11 \rightarrow 5x - 4 + 4 = 11 + 4 \rightarrow 5x = 15 \rightarrow 5x \div 5 = 15 \div 5 \rightarrow x = 3$
2. $-3a - 6 = 9 \rightarrow -3a - 6 + 6 = 9 + 6 \rightarrow -3a = 15 \rightarrow -3a \div -3 = 15 \div -3 \rightarrow a = -5$
3. $10 + 4c - 5 + 2c = -7 \rightarrow 5 + 6c = -7 \rightarrow 5 + 6c - 5 = -7 - 5 \rightarrow 6c = -12 \rightarrow 6c \div 6 = -12 \div 6 \rightarrow c = -2$
4. $\frac{x-4}{3} + 6 = -2 \rightarrow \frac{x-4}{3} + 6 - 6 = -2 - 6 \rightarrow \frac{x-4}{3} = -8 \rightarrow \frac{x-4}{3} \cdot 3 = -8 \cdot 3 \rightarrow x - 4 = -24 \rightarrow x - 4 + 4 = -24 + 4 \rightarrow x = -20$
5. $12x + 3 = 10x + 7 \rightarrow 12x + 3 - 10x = 10x + 7 - 10x \rightarrow 2x + 3 = 7 \rightarrow 2x + 3 - 3 = 7 - 3 \rightarrow 2x = 4 \rightarrow 2x \div 2 = 4 \div 2 \rightarrow x = 2$
6. $6 - 2x = 3x - 9 \rightarrow 6 - 2x + 2x = 3x - 9 + 2x \rightarrow 6 = 5x - 9 \rightarrow 6 + 9 = 5x - 9 + 9 \rightarrow 15 = 5x \rightarrow 3 = x$ or $x = 3$
7. $-6x + 8 = 2x \rightarrow -6x + 8 + 6x = 2x + 6x \rightarrow 8 = 8x \rightarrow 8 \div 8 = 8x \div 8 \rightarrow 1 = x$ or $x = 1$

8. $4(a - 5) = -24 \rightarrow 4a - 20 = -24 \rightarrow 4a - 20 + 20 = -24 + 20 \rightarrow 4a = -4 \rightarrow 4a \div 4 = -4 \div 4 \rightarrow a = -1$
9. $6b + 8 = 2(3b + 5) \rightarrow 6b + 8 = 6b + 10 \rightarrow 6b + 8 - 6b = 6b + 10 - 6b \rightarrow 8 = 10$. 8 can never equal 10, so this equation has no solution.
10. $6b - 24 = 3(2b - 8) \rightarrow 6b - 24 = 6b - 24 \rightarrow 6b - 24 - 6b = 6b - 24 - 6b \rightarrow -24 = -24$. –24 always equals –24, so this equation's solution is all the real numbers.
11. Let w = number of weeks. The equation is $125 + 7.50 \cdot w = 200 \rightarrow 125 + 7.50 \cdot w - 125 = 200 - 125 \rightarrow 7.50 \cdot w = 75 \rightarrow 7.50 \cdot w \div 7.50 = 75 \div 7.50 \rightarrow w = 10$. It will take 10 weeks.
12. Let x = the number. The equation is $8x = x + 35 \rightarrow 8x - x = x + 35 - x \rightarrow 7x = 35 \rightarrow 7x \div 7 = 35 \div 7 \rightarrow x = 5$
13. Let x = the number. The equation is $5x + 12 = 2x + 36 \rightarrow 5x + 12 - 2x = 2x + 36 - 2x \rightarrow 3x + 12 = 36 \rightarrow 3x + 12 - 12 = 36 - 12 \rightarrow 3x = 24 \rightarrow 3x \div 3 = 24 \div 3 \rightarrow x = 8$

Review

Hour's Up!

Based on what you've learned in Hour 6, solve the following equations. Check your answers with the solution key in Appendix B.

1. $6x + 3 = 15$
- **a.** $x = 3$
- **b.** $x = 4$
- **c.** $x = 1$
- **d.** $x = 2$

2. $-5a - 12 = 3$
- **a.** $a = 2$
- **b.** $a = -3$
- **c.** $a = 3$
- **d.** $a = -2$

3. $3x + 7x = 40$
- **a.** $x = 2$
- **b.** $x = 1$
- **c.** $x = 4$
- **d.** $x = 3$

Review

4. $6 + 5x - 8 + 2x = 19$
- **a.** $x = 3$
- **b.** $x = -3$
- **c.** $x = 2$
- **d.** $x = -2$

5. $\frac{x + 4}{5} + 3 = 9$
- **a.** $x = 2$
- **b.** $x = 26$
- **c.** $x = -4$
- **d.** $x = 1$

6. $6b + 7 = 4b + 9$
- **a.** $b = 1$
- **b.** $b = 2$
- **c.** $b = 3$
- **d.** $b = 4$

7. $4 - 3x = 5x - 12$
- **a.** $x = -2$
- **b.** $x = 2$
- **c.** $x = 1$
- **d.** $x = -1$

8. $-3\ (a + 2) = 2a + 4$
- **a.** $a = -2$
- **b.** $a = 2$
- **c.** $a = 1$
- **d.** $a = -1$

9. One car-rental company charges $42 per day with unlimited mileage. Another company charges $30 per day plus 15¢ per mile. You need to rent a car for a three-day trip. Find the break-even point, where the cost is the same for the two companies.
- **a.** 120 miles
- **b.** 200 miles
- **c.** 240 miles
- **d.** 300 miles

Review

10. Six times a number, increased by 20, is four less than eight times the number. Find the number.

a. 3
b. –4
c. –3
d. 12

Part III

All About Polynomials

Hour 7 Polynomials: Exponents and Basic Operations

Hour 8 Polynomials: Solving Equations and Multiplying

Hour 9 Polynomials: Dividing

Hour 10 Polynomials: Solving Equations and Factoring

Hour 7

Polynomials: Exponents and Basic Operations

Chapter Summary

In this hour you will learn about …

- Working with exponents on variable and numerical terms
- Following the order of operations for exponents
- Adding and subtracting polynomials
- Multiplying polynomials

You learned how to simplify algebraic expressions and solve equations, two important concepts that set algebra apart from basic arithmetic. The real-life problems you can now solve include car-rental cost comparisons and how long it takes to reach a savings goal. But you have been limited so far to one-dimensional problems. Many situations in life require two or three dimensions, from finding the square footage of a room when ordering new carpet, to calculating the cubic feet in your home to get the most efficient new furnace and air-conditioner.

Fully understanding algebra involves expanding your mind beyond a single dimension. From now on, many of the concepts are more intangible than the first six hours have been. Your goal is to be able to solve equations and work with expressions that are on a higher level. Taking it one hour at a time, building on the properties and processes you have learned so far, you can reach that goal.

Exponents

Think about the expression $3x + 4x$. To simplify it, you would write $3x + 4x = 7x$ because you know how to combine like terms. $7x$ is a very efficient way to describe $3x + 4x$. Similarly, algebra has efficient ways to simplify factors that are multiplied, as in the next example:

$x \cdot x \cdot x$ can be written x^3

Notice that three x factors are multiplied, and the new expression x^3 shows the x and a small 3 raised a little above the x. In x^3, the x is called the *base* and the 3 is called the *exponent* or *power*. Numbers can also be the base, as in the following example:

$8 \cdot 8 \cdot 8 \cdot 8$ can be written 8^4

STRICTLY DEFINED

An **exponent,** also called a **power,** is the small number that tells how many times the number below it (the **base**) is multiplied times itself. In a^5, 5 is the exponent and a is the base. a^5 is "a to the fifth power" or "a to the fifth."

As you see, multiplying 8 times itself four times (which is a very large number!) is very efficiently written with a base of 8 and an exponent of 4.

To say these expressions aloud, x^3 is "x to the third power" or shortened to "x to the third." You will also hear it described as "x cubed." 8^4 is "eight to the fourth power" or simply "eight to the fourth."

An exponent of 2 is common. b^2 is "b to the second power," "b to the second," or "b squared." A variable or number with no exponent showing can be written with an exponent of 1. So $x = x^1$, and you can say, "x is equal to x to the first power," and $5 = 5^1$, or "5 equals 5 to the first power."

The exponent tells you the number of times the base is multiplied by itself. The following examples show the expanded and exponential forms of several expressions:

JUST A MINUTE

Remember that parentheses with no other operation indicate multiplication. They can also be used simply for emphasis, as in example h.

a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot = 2^5$
b. $c \cdot c \cdot c \cdot = c^3$
c. $(4)(4)(5)(5)(5) = 4^2 \cdot 5^3$
d. $7^4 = 7 \cdot 7 \cdot 7 \cdot 7$
e. $x^3 \cdot y^2 = x \cdot x \cdot x \cdot y \cdot y$
f. $5a^4b^3c = 5 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c$
g. $-3x^5 = -3 \cdot x \cdot x \cdot x \cdot x \cdot x$
h. $6 \cdot x \cdot (-3) \cdot x \cdot x = -18x^3$
i. $-2 \cdot a \cdot a \cdot -4 \cdot a \cdot b \cdot b = 8a^3b^2$

Now write the following expressions in expanded or exponential form, checking your answers on page 84.

1. $9 \cdot 9 \cdot 9 \cdot 9 =$
2. $-6cd^4 =$
3. $5 \cdot 5 \cdot 5 \cdot x \cdot x \cdot y =$
4. $3 \cdot a \cdot (-3) \cdot b \cdot b =$

Following the Order of Operations with Exponents

When algebraic expressions have more than one operation, you learned in Hour 1 that terms in parentheses are combined first. When exponents are included in an expression, they are next in the order, followed by multiplying and dividing from left to right. Adding and subtracting is then done from left to right.

The following examples show this sequence. Note especially what happens when parentheses are around different terms in these problems. The answers vary greatly, and it's all due to where the parentheses are.

j. $2 \cdot 3^2 = 2 \cdot 9 = 18$
k. $(2 \cdot 3)^2 = (6)^2 = 6 \cdot 6 = 36$
l. $2 + 3^2 = 2 + 9 = 11$
m. $(2 + 3)^2 = (5)^2 = 25$
n. $2^2 + 3^2 = 4 + 9 = 13$
o. $2^2 + 3 = 4 + 3 = 7$
p. $(-3)^2 = (-3) \cdot (-3) = 9$
q. $-3^2 = -(3 \cdot 3) = -9$
r. $(3x)^2 = (3x)(3x) = 9x^2$
s. $3x^2 = 3 \cdot x \cdot x$

FYI Note carefully the importance of the parentheses in example r: the 3 inside the parentheses is squared to make 9. In example s, the 3 is not affected by the exponent 2—only the *x* is squared.

Many formulas used in medicine, finance, physics, and engineering include exponents. Being able to evaluate a formula when you are given values for the variables is the most common real-life application of exponents. The following examples show the correct substitutions and order of operations.

t. If $a = 3$ and $b = 4$, evaluate ab^2.

Solution: $3 \cdot 4^2 = 3 \cdot 16 = 48$

u. If $a = 3$ and $b = 4$, evaluate $(ab)^2$.

Solution: $(3 \cdot 4)^2 = (12)^2 = 12 \cdot 12 = 144$

v. If $x = 4$ and $y = 3$, evaluate $x^2 - y^2$.

Solution: $4^2 - 3^2 = 16 - 9 = 7$

w. If $a = -2$ and $b = 5$, evaluate $3a^2 + 2b^2$.

Solution: $3(-2)^2 + 2(5)^2 = 3 \cdot -2 \cdot -2 + 2 \cdot 5 \cdot 5 = 12 + 50 = 62$

x. If $x = 2$ and $y = 3$, evaluate $\dfrac{(x + y)^2}{2x - y}$.

Solution: $\dfrac{(2 + 3)^2}{2 \cdot 2 - 3} = \dfrac{5^2}{4 - 3} = \dfrac{25}{1} = 25$

Now try the following problems. Check your answers on page 84.

5. $2 \cdot 4^2 =$
6. $(2 \cdot 4)^2 =$
7. $2 + 4^2 =$
8. $(2 + 4)^2 =$
9. $-5 + (-2)^3 =$
10. If $x = 2$ and $y = 4$, evaluate $(x + y)^2 - x^2 + y^2$.
11. If $a = -3$ and $b = 5$, evaluate $a + b + \dfrac{(a - b)^2}{a + b}$.

It's time to define the key word in this hour's title, *polynomials*. In fact, you've been working with polynomials all hour! Instead of learning a formal definition, look at the following examples to get a working knowledge of some big words that describe simple ideas.

STRICTLY DEFINED

Polynomials are sums of expressions that can include numerals, variables, and exponents in the terms. Monomials have only one term, binomials have two terms, and trinomials have three terms. The exponents must be whole numbers (no fractions or negative numbers), and polynomials cannot have any variables in the denominator. Therefore, $(x + y) \div (2x - y)$ is not an example of a polynomial, but it is an algebraic expression that can be evaluated if you know the x and y values.

In this first example, c^{32} and $4a^2$ are each a kind of polynomial called *monomials* because they don't include any adding or subtracting, only multiplication. $2x^4 + 5$ and $3x^2 - 7x$ are both called *binomials* because they are made up of two monomial terms that are added or subtracted. A polynomial like $2x^2 - 3x + 12$ is called a *trinomial* because there are three monomial terms making one expression. Expressions with four or more terms are simply called polynomials.

The following table has examples of algebraic expressions and their descriptive names:

Expression	*Number of Terms*	*Descriptive Name*
$8x^2y^3$	One	Monomial
$5a^4 + 7a$	Two	Binomial
$-3b^3 + 2b - 7$	Three	Trinomial
$2x^3 - 3x^2 - 5x + 8$	Four	Polynomial
$\frac{9}{x}$	One	Not a polynomial
$4a^{-2} + 6a - 5$	Three	Not a polynomial

Simplifying polynomials follows the same rules you used last hour to solve equations. You already know how to simplify $7x + 2x + 5$ to get $9x + 5$. The $7x$ and the $2x$ terms can be added together because they are like terms—they both have the same variable x. The coefficients 7 and 2 sum to 9, so the result is $9x + 5$.

Similarly, $7x^2 + 2x^2 = 9x^2$ because $7x^2$ and $2x^2$ are like terms. Notice, though, that the exponent didn't change to a 4, and this is important: when like terms are added or subtracted, the exponent never changes. Like terms must have the same base and exponent, but their coefficients do not need to be the same.

JUST A MINUTE

Remember that the coefficient of a variable is the number in front of the variable, and if no number is shown, it is understood to be a 1. If the coefficients of two terms sum to zero, the entire term becomes a zero: $-3x + 3x = 0 \cdot x = 0$. Note that the –3 and +3 (the coefficients of the variables x) sum to zero, so the entire term becomes a zero.

The following examples show how to simplify polynomials by first grouping the like terms and then combining them.

Simplify:

y. $6x^2 + 5x - 2x^2 + 3x = 6x^2 - 2x^2 + 5x + 3x = 4x^2 + 8x$
z. $3a^3 + 7a - 2 - a^3 + 7a = 3a^3 - a^3 + 7a + 7a - 2 = 2a^3 + 14a - 2$

Now try simplifying the following polynomials. Check your answers on page 84.

12. $x^4 - 3x^3 + 4x^4 + 2x^3 + 4 =$
13. $2a^2 + 6b^3 - 5a^2 - 4 + 2b^3 =$
14. $3y + 3y^2 + 5z + 3y =$
15. $9xy - 3x^3 + 6x^3 - 9xy =$

Adding and Subtracting Polynomials

The simplifying you did in problems 12 through 15 leads directly to learning how to add or subtract polynomials. You still need to look for like terms. The problems look a bit different than the simplifying problems do, but the addition and subtraction examples that follow show how easy the process is:

Add $2x^3 - 3x^2 + x + 5$ and $8x^3 + 3x^2 + x - 7$.

This problem can also be written as $(2x^3 - 3x^2 + x + 5) + (8x^3 + 3x^2 + x - 7)$.

Solution: These two polynomials can be stacked vertically, or the similar terms in each can be grouped, as in examples w through z. The vertical method is shown first:

$$\begin{array}{r} 2x^3 - 3x^2 + x + 5 \\ 8x^3 + 3x^2 + x - 7 \\ \hline 10x^3 \qquad + 2x - 2 \end{array}$$

The grouping method is as follows:

$$2x^3 + 8x^3 - 3x^2 + 3x^2 + x + x + 5 - 7 = 10x^3 + 2x - 2$$

Subtract $2x^3 - 3x^2 + x + 5$ from $8x^3 + 3x^2 + x - 7$.

Solution: Remember that subtraction must be done in the proper order. For example, to subtract 3 from 7, use $7 - 3 = 4$, not $3 - 7$ (which is -4). To subtract the first polynomial from the second one, be sure the second one is put first, before the subtraction sign.

This problem can also be written as follows:

$(8x^3 + 3x^2 + x - 7) - (2x^3 - 3x^2 + x + 5)$

The subtraction sign becomes addition, but you must change each sign in the polynomial being subtracted, which is shown here:

$$\begin{array}{r} 8x^3 + 3x^2 + x - 7 \\ +\ -2x^3 + 3x^2 - x - 5 \\ \hline 6x^3 + 6x^2 - 12 \end{array}$$

Subtraction can also be done by the grouping method:

$(8x^3 + 3x^2 + x - 7) - (2x^3 - 3x^2 + x + 5)$ becomes an addition problem if you change all the signs in the second polynomial:

$$(8x^3 + 3x^2 + x - 7) + (-2x^3 + 3x^2 - x - 5)$$

The like terms can now be grouped and combined:

$$8x^3 - 2x^3 + 3x^2 + 3x^2 + x - x - 7 - 5 = 6x^3 + 6x^2 - 12$$

As you can see, subtraction is more complex than addition, but following the process carefully and watching every sign as you work is the key to success with these problems.

Now try adding and subtracting the following polynomials. Check your answers on pages 84–85.

16. Add $7x + 5$ and $2x - 4$.
17. Add $5a^3 + 8a^2 - 6$ and $2a^3 + 2a^2 + 6$.
18. $(3x^4 - 4x^2 + 12) + (x^4 + 5x^2 + 2) =$
19. Subtract $a^2 - 6a + 9$ from $7a^2 - 2a + 8$.
20. $(x^2 - 5x + 8) - (7x^2 + 5x + 2) =$

GO TO ◄
Subtraction can be changed to addition if you change the signs of all the terms in the second expression. (See Hour 3.)

GO TO ◄
Refer to the sign rules in Hour 3 for adding positive and negative numbers when combining like terms.

Multiplying Exponential Terms

This hour began with the following example: $x \cdot x \cdot x = x^3$.

It shows both the expanded form and the exponential form of "x to the third power." Perhaps because exponents are a shortcut way to show multiplication of a base term, learning how to multiply monomials is actually easier than the adding and subtracting section you just completed.

Consider this multiplication problem:

$$x^2 \cdot x^3$$

If you expand each of the terms, it becomes $x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x$, which equals x^5.

To multiply $a^4 \cdot a^3$, expanding each term results in $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$, which equals a^7.

After a few of these examples, you surely can see the general rule for multiplying terms that have the same base: you simply keep the base and add the exponents.

Does this work for several monomials? Yes, if the bases are the same.

The following examples show how this rule is applied. The expanded form is shown here as explanation, but it is not needed. Notice also that the coefficients of the monomials are multiplied.

TIME SAVER

The exponential form shown for examples A through F helps you see how each of the monomials can be written as factors, but you should be able to give the answer without showing this step.

A. $b^5 \cdot b^2 = b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b = b^7$

B. $(2x^2)(5x^4) = 2 \cdot x \cdot x \cdot 5 \cdot x \cdot x \cdot x \cdot x = 10x^6$

C. $(4a^3)(-6a) = 4 \cdot a \cdot a \cdot a \cdot -6 \cdot a = -24a^4$

D. $(-5x^2y)(2xy^3)(xy) = -5 \cdot x \cdot x \cdot y \cdot 2 \cdot x \cdot y \cdot y \cdot y \cdot x \cdot y = -10x^4y^5$

E. $\dfrac{12a^3b^2}{5} \cdot \dfrac{10ab^4}{3} = \dfrac{12}{5} \cdot \dfrac{10}{3} \cdot a \cdot a \cdot a \cdot b \cdot b \cdot a \cdot b \cdot b \cdot b \cdot b = \cdot$

$\dfrac{4 \cdot 3}{5} \cdot \dfrac{5 \cdot 2}{3} \cdot a^4b^6 = 8a^4b^6$

F. $(2xy^2)(5xy^3) + (6y^4)(x^2y) = 2 \cdot x \cdot y \cdot y \cdot 5 \cdot x \cdot y \cdot y \cdot y + 6 \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot y = 10x^2y^5 + 6x^2y^5 = 16x^2y^5$

Look again at examples B through F, and try to skip the expansion step to get from the problem directly to the solution by simply adding the exponents of the equal bases. Now that you have seen the detailed work on these problems, your goal should be to accurately move from the problem to the solution without expanding the terms with exponents, by adding the exponents instead.

As you reach the end of this hour on exponents, the final section explains how to simplify terms with exponents that are raised to another exponent. Expanding the terms, though, shows that the process is surprisingly easy to learn, and you'll probably see a shortcut.

PROCEED WITH CAUTION

The parentheses in these "power" problems show that the factors inside will be multiplied times themselves, so they are important to each of the problems and are not just for emphasis.

Simplify: $(x^3)^2$

This problem could be described as "x to the third power, raised to the second power" or "x to the third, times itself." To simplify it, expand it to see the exponents in clearer form:

$$(x^3)^2 = (x^3) \cdot (x^3) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$$

Simplify: $(2b^2)^3$

This could be described as "2 b squared, raised to the third power." Expand it to see the factors more clearly:

$$(2b^2)^3 = (2b^2)(2b^2)(2b^2) = 2 \cdot b \cdot b \cdot 2 \cdot b \cdot b \cdot 2 \cdot b \cdot b = 8b^6$$

The parentheses continue to play a very important role, so pay attention to what is inside them. Numbers or variables outside the parentheses are not affected by the exponent, but numbers or variables inside are affected, as you see in the following examples:

Simplify: $5x(y^4)^2$

Solution: $5x(y^4)^2 = 5x \cdot y^4 \cdot y^4 = 5x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y = 5xy^8$

Simplify: $(5xy^4)^2$

Solution: $(5xy^4)^2 = 5xy^4 \cdot 5xy^4 = 5 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y = 25x^2y^8$

Simplify: $-6(a^4b^3)^2$

Solution: $-6(a^4b^3)^2 = -6 \cdot a^4b^3 \cdot a^4b^3 = -6 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b = -6a^8b^6$

Simplify: $(-6a^4b^3)^2$ (Try to skip the expansion step.)

Solution: $(-6a^4b^3)^2 = -6a^4b^3 \cdot -6a^4b^3 = -6 \cdot -6 \cdot a^4 \cdot a^4 \cdot b^3 \cdot b^3 = 36a^8b^6$

TIME SAVER

The rule for raising a power to a power: multiply the powers.

Now try simplifying the following polynomials. Check your answers on page 85.

21. $a^4 \cdot a^5 =$
22. $(-3c^3)(6c^2) =$
23. $(2x^3y^2)(-3xy)(-2xy^4) =$
24. $(3a^2b^3)(4ab^2) + (2b^4)(a^3b) =$
25. $(a^4)^5 =$
26. $(4a^3b^4)^2 =$
27. $4(a^3b^4)^2 =$

Answers to Sample Problems

1. 9^4
2. $-6 \cdot c \cdot d \cdot d \cdot d \cdot d$
3. 5^3x^2y
4. $-9ab^2$
5. $2 \cdot 16 = 32$
6. $(8)^2 = 64$
7. $2 + 16 = 18$
8. $6^2 = 36$
9. $-5 + -2 \cdot -2 \cdot -2 = -5 + -8 = -13$
10. $(2 + 4)^2 - 2^2 + 4^2 = (6)^2 - 4 + 16 = 36 - 4 + 16 = 32 + 16 = 48$
11. $-3 + 5 + \dfrac{(-3-5)^2}{-3+5} = -3 + 5 + \dfrac{(-8)^2}{2} = -3 + 5 + \dfrac{64}{2} = -3 + 5 + 32 = 2 + 32 = 34$
12. $5x^4 - x^3 + 4$
13. $-3a^2 + 8b^3 - 4$
14. $6y + 3y^2 + 5z$
15. $3x^3$
16. $7x + 5 + 2x - 4 = 9x + 1$
17. $5a^3 + 8a^2 - 6 + 2a^3 + 2a^2 + 6 = 7a^3 + 10a^2$

18. $4x^4 + x^2 + 14$
19. $7a^2 - 2a + 8 - (a^2 - 6a + 9) = 7a^2 - 2a + 8 - a^2 + 6a - 9 = 6a^2 + 4a - 1$
20. $(x^2 - 5x + 8) - (7x^2 + 5x + 2) - x^2 - 5x + 8 - 7x^2 - 5x - 2 = -6x^2 - 10x + 6$
21. a^9
22. $-18c^5$
23. $12x^5y^7$
24. $(3a^2b^3)(4ab^2) + (2b^4)(a^3b) = 12a^3b^5 + 2a^3b^5 = 14a^3b^5$
25. a^{20}
26. $16a^6b^8$
27. $4a^6b^8$

Hour's Up!

Based on what you've learned in Hour 7, do the following problems. Check your answers with the solution key in Appendix B.

1. Simplify: $4 \cdot 3^2 =$
- **a.** 144
- **b.** 49
- **c.** 12
- **d.** 36

2. Simplify: $(4 \cdot 3)^2 =$
- **a.** 144
- **b.** 49
- **c.** 13
- **d.** 36

3. Simplify: $4 + 3^2 =$
- **a.** 49
- **b.** 10
- **c.** 13
- **d.** 4

4. Simplify: $(2a)^3 =$
- **a.** $6a$
- **b.** $2a^3$
- **c.** $6a^3$
- **d.** $8a^3$

5. If $x = -4$ and $y = 3$, evaluate $x - y^2 + \frac{(x - y)^2}{x + y}$.
a. -62
b. 54
c. -44
d. -50

6. Simplify: $7a^2b + 5b^3 - 5a^2b - 4 + 3b^3 =$
a. $12a^2b + 8b^3 - 4$
b. $2a^2b + 8b^3 - 4$
c. $2a^2b + 2b^3$
d. $2a^2b + 2b^3 - 4$

7. Simplify: $(8x^4 - 3x^2 + 9) + (x^4 + 6x^2 + 4) =$
a. $9x^8 + 3x^4 + 13$
b. $8x^4 + 3x^2 + 13$
c. $9x^4 + 3x^2 + 13$
d. $8x^4 + 9x^2 + 13$

8. Simplify: $(5a^3 + 4a^2 - 2a + 6) - (a^3 + 6a^2 - 2a) =$
a. $5a^3 + 10a^2 + 6$
b. $4a^3 - 2a^2 + 6$
c. $4a^3 - 2a^2 - 6$
d. $5a^3 - 2a^2 + 6$

9. Simplify: $(2x^3y)(4xy^4) + (6y^2)(x^4y^3) =$
a. $48x^8y^{10}$
b. $14x^8y^{10}$
c. $14x^4y^5$
d. $14x^8y^{10}$

10. Simplify: $(6a^4b^5)^2 =$
a. $36a^8b^{10}$
b. $12a^6b^7$
c. $12a^8b^{10}$
d. $6a^6b^7$

Review

Hour 8

Polynomials: Solving Equations and Multiplying

CHAPTER SUMMARY

In this hour you will learn about …

- Solving equations that include polynomial terms
- Solving word problems with basic polynomial equations
- Multiplying monomials and polynomials
- Multiplying two binomials using the FOIL method

The exponents that you learned about in the previous hour are powerful—no pun intended—and they allow you to work with algebra problems that are quite complex. Don't forget that they are efficient, as they help make lengthy multiplication expressions very compact.

You can now add and subtract polynomials that include exponents. You can also multiply monomials to get a simplified answer and raise monomials to powers. In this hour, you learn about multiplying polynomials by monomials and by other polynomials, and how polynomials can be used to solve word problems. First, let's combine your knowledge of polynomials with basic equation skills to solve some new equations.

Solving Equations That Include Polynomial Terms

Look at the following equation that includes a difference of two binomials:

a. $(3x + 4) - (x + 1) = 13$

To solve it, simplify the left side by subtracting the binomials and then isolate the variable.

JUST A MINUTE

Subtraction can be changed to addition if you change the sign of the terms after the subtraction sign. For example, $2 - 6 = 2 + -6 = -4$ and $9 - (x + 5) = 9 + -x + -5 = 4 - x$.

Solve: $(3x + 4) - (x + 1) = 13$

$3x + 4 + -x + -1 = 13$
$3x + -x + 4 + -1 = 13$
$2x + 3 = 13$
$2x + 3 - 3 = 13 - 3$
$2x = 10$
$2x \div 2 = 10 \div 2$
$x = 5$

To check, put 5 in for x in the original equation:

$(3 \cdot 5 + 4) - (5 + 1) = 13 \rightarrow 3 \cdot 5 + 4 - (6) = 13 \rightarrow 15 + 4 - 6 = 13 \rightarrow 19 - 6 = 13 \rightarrow 13 = 13$. Therefore, $x = 5$ is the solution.

With your ability to simplify polynomials that are added or subtracted, you can solve equations that contain these expressions. They just need to be simplified to become the same types of equations you learned how to solve in Hours 5 and 6.

Now try to solve the following equations, and check your answers on page 96.

1. $(5a + 6) + (2a - 7) = 6$
2. $12x + (6 - 2x) = 4(x - 3)$
3. $2(a + 4) - 5(a + 1) = 15$
4. $(b - 8) + (3b + 7) = (b - 3) + (b + 12)$

Solving Word Problems with Basic Polynomial Equations

Polynomials appear in many different kinds of word problems that can be written as equations and then solved. Some, such as problems about consecutive integers, are totally based on algebra. Others, like distance problems, come from more practical situations. As in Hours 1 and 6, the first step is to change the word problem into an algebraic equation. Reading each problem carefully will help you decide what the variable is and what operation is needed.

JUST A MINUTE

Remember that consecutive integers are next to each other on the number line. Given any integer, add 1 to find the consecutive integer. For example, 6 and 7 are consecutive integers, as are –10 and –9. If the variable **x** is an integer, the consecutive integer is $x + 1$.

Consider the following problem about consecutive integers:

b. What two consecutive integers have a sum of 47?

Solution: Begin by writing an equation that describes the problem. Let x be the first integer, so $x + 1$ is the consecutive integer. Write and solve the equation:

$$x + (x + 1) = 47$$
$$2x + 1 = 47$$
$$2x + 1 - 1 = 47 - 1$$
$$2x = 46$$
$$2x \div 2 = 46 \div 2$$
$$x = 23$$

To check, put $x = 23$ into the original equation:

$$23 + (23 + 1) = 47 \rightarrow 23 + 24 = 47 \rightarrow 47 = 47$$

Therefore, $x = 23$ is the solution. The question asks for two consecutive integers, and they are 23 and 24.

The next two problems have operations that are slightly more complex:

c. The greater of two consecutive integers is 15 less than twice the smaller integer. What are the integers?

Solution: Let x and $x + 1$ be the consecutive integers, as in the previous problem. To write the equation, you need to know which is the greater, x or $x + 1$. Adding 1 to a number always gives a number that's greater, so $x + 1$ is the greater integer. Begin with $x + 1$ as you change the written problem into an equation. "Less than" indicates subtraction, so be careful to put the terms in the correct order. Write and solve the equation:

$$x + 1 = 2x - 15$$
$$x + 1 - x = 2x - 15 - x$$
$$1 = x - 15$$
$$1 + 15 = x - 15 + 15$$
$$16 = x \text{ or } x = 16$$

To check, put $x = 16$ into the original equation:

$$16 + 1 = 2 \cdot 16 - 15 \rightarrow 17 = 32 - 15 \rightarrow 17 = 17$$

Therefore, $x = 16$ is the solution. The integers are 16 and 17.

As with the prior problem, the next one includes more complex operations.

d. Twice the greater of two consecutive integers is 7 less than three times the smaller integer. What are the integers?

Solution: Again, x and $x + 1$ will represent the consecutive integers, with $x + 1$ being the greater one. Translate the words of the problem into an equation and solve it. "Twice" means multiplying by 2, so the first part of the problem is written as $2(x + 1)$. "7 less than three times the smaller integer" is written as $3x - 7$. (Remember that subtraction is the "less than" operation, and the order is important in subtraction.) Figure out these algebraic phrases together to write your equation, as shown here.

$$2(x + 1) = 3x - 7$$
$$2x + 2 = 3x - 7$$
$$2x + 2 - 2x = 3x - 7 - 2x$$
$$2 = x - 7$$
$$2 + 7 = x - 7 + 7$$
$$9 = x \text{ or } x = 9$$

To check, put $x = 9$ into the original equation:

$$2(9 + 1) = 3 \cdot 9 - 7 \rightarrow 2 \cdot 10 = 27 - 7 \rightarrow 20 = 20$$

Therefore, $x = 9$ is the solution, and the two consecutive integers are 9 and 10.

Now try to solve the following word problems, and check your answers on page 96.

5. What two consecutive integers have a sum of 37?
6. Three times the greater of two consecutive integers is 5 less than four times the smaller integer. What are the integers?

These consecutive integer word problems bring together basic monomials, such as x and $3x$, and basic polynomials, such as $2x + 2$, creating equations that can be solved. The rest of this hour, you learn how to multiply and simplify more complex monomials and polynomials, using the distributive property from Hour 4 and your knowledge of exponents from Hour 7.

Multiplying Monomials and Polynomials

The distributive property allows you to simplify expressions that have a binomial multiplied by a number. To put it more plainly, you know that, to simplify $3(x + 7)$, you multiply 3 times x and 3 times 7. The answer, $3x + 21$, is called "simplified" because the answer has no parentheses and is now a simple binomial. This same property works when monomials and polynomials are multiplied, as in the following examples.

JUST A MINUTE

The distributive property says that when you multiply a sum by a number, it is the same as adding the product of the numbers. For example, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

e. Simplify: $a(a + 8)$

Solution: Multiply a times each term inside the parentheses.

$$a(a + 8) = a^2 + 8a$$

To multiply terms that have the same base, keep the base and add the exponents. For example, $(x^2)(x^3) = x^{2+3} = x^5$. If a factor has a coefficient, multiply the coefficients of each factor. For example, $(3x)(4x) = 12x^2$. Terms that show no coefficient have a coefficient of 1. The following examples show how to work with the distributive property that includes terms with exponents.

f. Simplify: $6x(2x^2 + 3x - 4)$

Solution: The distributive property is not limited to binomials. Any number of terms can be included within the parentheses. To simplify, multiply $6x$ times each of the trinomial's terms, following the rules for exponents that you learned in Hour 7:

$$6x(2x^2 + 3x - 4) =$$
$$6x \cdot 2x^2 + 6x \cdot 3x + 6x \cdot -4$$
$$12x^3 + 18x^2 - 24x$$

g. Simplify: $-3a^2b(2a^3 + 5a^2b^2 - 3b^3)$

Solution: To simplify, multiply $-3a^2b$ times each of the trinomial's terms. Watch the signs very carefully!

$-3a^2b(2a^3 + 5a^2b^2 - 3b^3) =$
$-3a^2b \cdot 2a^3 - 3a^2b \cdot 5a^2b^2 - 3a^2b \cdot -3b^3 =$
$-6a^5b - 15a^4b^3 + 9a^2b^4$

Now try these problems, using the distributive property, order of operations, and exponent rules to combine the like terms, as needed. Check your answers on page 96.

7. $5x(2x^2 + 3x - 5) =$
8. $-2a^4(3a^2 - 2a + 6) =$
9. $3x(x^2 + 4x - 1) + 4x(2x^2 + 5x + 4) =$
10. $a^2(3a - 4) - a(2a^2 + 5a) =$

As you have seen throughout this hour, the process of multiplying monomials times other polynomials brings together several different algebraic skills. It is a perfect example of what you may have heard about algebra: each new concept builds on the previous one. This logical progression leads to looking at what happens when you multiply by a binomial instead of a monomial.

Multiplying Polynomials

Now that you can multiply by monomials, let's push the distributive property to the next level and learn how to multiply by a binomial expression.

You know how to do the following multiplication problem, but look closely at the solution steps:

$$x(x^2 + 3) = x \cdot x^2 + x \cdot 3 = x^3 + 3x$$

Using the distributive property, you multiply the x in front of the parentheses times each of the terms inside the parentheses—$x^2 + 3$—as your first step and then simplify the terms to get the answer.

Now change the x in front of the parentheses to $(x + 2)$.

The new problem is:

$$(x + 2)(x^2 + 3)$$

To simplify it, follow the same process. Multiply $(x + 2)$ times each of the terms inside the parentheses as your first step. Then simplify to get the answer. The steps look like this:

$(x + 2)(x^2 + 3)$
$(x + 2) \cdot x^2 + (x + 2) \cdot 3 =$
$x^3 + 2x^2 + 3x + 6$

JUST A MINUTE

The distributive property applies when the terms are reversed: $(b + c) \cdot a = a \cdot b + a \cdot c$

Be sure to check for like terms to see if more work is needed. There are no like terms, so this is the final answer.

You can always use the distributive property to simplify polynomial multiplication problems like this one, but there's a shortcut that is a little easier. Look at how the same problem can be done with another method:

$(x + 2)(x^2 + 3) =$
$x \cdot x^2 + x \cdot 3 + 2 \cdot x^2 + 2 \cdot 3 =$
$x^3 + 3x + 2x^2 + 6$

Comparing this answer to the previous one, you see that they are the same; the terms of a polynomial can be written in any order. Usually the answer is shown in descending order, with the highest exponent first, but this is not required.

This shortcut method can be described as follows:

> Multiply each of the terms in the first set of parentheses times each term in the second set. This will have the same effect as applying the distributive property, but it's easier to remember and seems easier than going through the distributive process. You are free to use either method.

This multiplication process can be used for more than just binomials, as you see in the next example of multiplying a binomial times a trinomial:

$(x + 5)(x^2 + 4x - 6) =$
$x \cdot x^2 + x \cdot 4x + x \cdot -6 + 5 \cdot x^2 + 5 \cdot 4x + 5 \cdot -6 =$
$x^3 + 4x^2 - 6x + 5x^2 + 20x - 30 =$
$x^3 + 9x^2 + 14x - 30$

PROCEED WITH CAUTION

Be careful to combine only like terms, which are those with exactly the same variables and exactly the same exponents. When adding and subtracting terms, the exponents can never change—only the coefficients will change.

You do not need to write the step after the first equals sign. It is shown here to help you see the process at work. You should be able to omit that line of work and go directly to the line that begins with $x^3 + 4x^2 - 6x$ …. Be sure that you watch the signs in the polynomial terms and combine all the like terms to get the final answer in simplest form.

Now try these multiplication problems, writing your final answer in simplest form. Check your answers on pages 96–97.

11. $(x + 3)(x + 5) =$
12. $(x + 2)(x^2 + 3x - 4) =$

Multiplying two binomials is the most common type of polynomial multiplication in algebra, so let's finish the hour learning an efficient method.

Multiplying Two Binomials

When you worked on problem 11 in the previous section, you probably multiplied x times $x + 5$ and 3 times $x + 5$, and then combined the like terms. There's an even shorter way to multiply two simple binomials like these. See it develop as you look at two more examples of binomial multiplication:

$$(x + 4)(x + 6) =$$
$$x^2 + 6x + 4x + 24 =$$
$$x^2 + 10x + 24$$

and

$$(x + 3)(x + 7) =$$
$$x^2 + 7x + 3x + 21 =$$
$$x^2 + 10x + 21$$

The basic idea for the shortcut is to give special labels to the terms in each binomial based on its location in the problem. Using $(x + 2)(x + 4)$ as the example, label the terms as follows:

First terms	x and x	(the first term in each binomial)
Outside terms	x and 4	(the "outer" terms of the problem)
Inside terms	2 and x	(the "inner" terms of the problem)
Last terms	2 and 4	(the last term in each binomial)

Multiplying the first terms—x and x— $\rightarrow x^2$
Multiplying the outer terms—x and 4— $\rightarrow + 4x$
Multiplying the inner terms—2 and x— $\rightarrow + 2x$
Multiplying the last terms—2 and 4— $\rightarrow + 8$
Combining them all gives $x^2 + 6x + 8$.

This is called the FOIL method, based on the first letters of the identified terms—First, Outside, Inside, Last. It also works if any of the terms are negative. You just need to pay close attention to the signs. After you FOIL a few problems, you will see how efficient it is.

JUST A MINUTE

Remember that subtracting is the same as adding the opposite: $x - 4 = x + (-4)$.

Follow the FOIL steps for $(x + 6)(x + 3)$, shown here on one line:

$(x + 6)(x + 3) \rightarrow$ First: x^2 Outside: $+ 3x$ Inside: $+ 6x$ Last: $+ 18$

Combining all the terms, you get $\rightarrow x^2 + 9x + 18$.

The following example includes a negative term:

$(x - 4)(x + 5) \rightarrow$ First: x^2 Outside: $+ 5x$ Inside: $-4x$ Last: -20

Combining all the terms, you get $\rightarrow x^2 + x - 20$.

This final example shows the unique case of the disappearing x term:

$(x + 6)(x - 6) \rightarrow$ First: x^2 Outside: $- 6x$ Inside: $+ 6x$ Last: -36

Combining all the terms, you get $\rightarrow x^2 - 36$. (Note: Because $6x$ and $-6x$ are opposites, the sum of these terms is $0x$ or just 0.)

You get this type of answer when the last terms in the binomials are opposites of each other, as in:

$$(x - 5)(x + 5) = x^2 + 5x - 5x - 25 = x^2 - 25$$

You can skip the middle step or show the work, as you prefer.

Now try these multiplication problems, using the FOIL method:

13. $(x + 2)(x + 5) =$
14. $(x + 8)(x + 1) =$
15. $(x - 3)(x + 4) =$
16. $(x + 3)(x - 3) =$

Check your answers on page 97.

Answers to Sample Problems

1. $7a - 1 = 6 \rightarrow 7a - 1 + 1 = 6 + 1 \rightarrow 7a = 7 \rightarrow 7a \div 7 = 7 \div 7 \rightarrow a = 1$
2. $12x + 6 - 2x = 4x - 12 \rightarrow 10x + 6 = 4x - 12 \rightarrow 10x + 6 - 4x = 4x - 12 - 4x \rightarrow 6x + 6 = -12 \rightarrow 6x + 6 - 6 = -12 - 6 \rightarrow 6x = -18 \rightarrow 6x \div 6 = -18 \div 6 \rightarrow x = -3$
3. $2a + 8 - 5a - 5 = 15 \rightarrow -3a + 3 = 15 \rightarrow -3a + 3 - 3 = 15 - 3 \rightarrow -3a = 12 \rightarrow -3a \div -3 = 12 \div -3 \rightarrow a = -4$
4. $4b - 1 = 2b + 9 \rightarrow 4b - 1 - 2b = 2b + 9 - 2b \rightarrow 2b - 1 = 9 \rightarrow 2b - 1 + 1 = 9 + 1 \rightarrow 2b = 10 \rightarrow 2b \div 2 = 10 \div 2 \rightarrow b = 5$
5. The equation is $x + (x + 1) = 37 \rightarrow 2x + 1 = 37 \rightarrow 2x + 1 - 1 = 37 - 1 \rightarrow 2x = 36 \rightarrow 2x \div 2 = 36 \div 2 \rightarrow x = 18$. The two integers are 18 and 19.
6. The equation is $3(x + 1) = 4x - 5 \rightarrow 3x + 3 = 4x - 5 \rightarrow 3x + 3 - 3x = 4x - 5 - 3x \rightarrow 3 = x - 5 \rightarrow 3 + 5 = x - 5 + 5 \rightarrow 8 = x$ or $x = 8$. The two integers are 8 and 9.
7. $5x(2x^2 + 3x - 5) = 10x^3 + 15x^2 - 25x$
8. $-2a^4(3a^2 - 2a + 6) = -6a^6 + 4a^5 - 12a^4$
9. $3x(x^2 + 4x - 1) + 4x(2x^2 + 5x + 4) =$
   $3x^3 + 12x^2 - 3x + 8x^3 + 20x^2 + 16x =$
   $11x^3 + 32x^2 + 13x$
10. $a^2(3a - 4) - a(2a + 5a) =$
    $3a^3 - 4a^2 - 2a^3 - 5a^2 =$
    $a^3 - 9a^2$
11. $(x + 3)(x + 5) =$
    $x \cdot x + x \cdot 5 + 3 \cdot x + 3 \cdot 5$
    $x^2 + 5x + 3x + 15 =$
    $x^2 + 8x + 15$

12. $(x + 2)(x^2 + 3x - 4) =$
$x^3 + 3x^2 - 4x + 2x^2 + 6x - 8 =$
$x^3 + 5x^2 + 2x - 8$
13. $(x + 2)(x + 5) =$
$x^2 + 5x + 2x + 10 =$
$x^2 + 7x + 10$
14. $(x + 8)(x + 1) =$
$x^2 + 1x + 8x + 8 =$
$x^2 + 9x + 8$
15. $(x - 3)(x + 4) =$
$x^2 + 4x - 3x - 12 =$
$x^2 + x - 12$
16. $(x + 3)(x - 3) =$
$x^2 - 3x + 3x - 9 =$
$x^2 - 9$

Hour's Up!

Based on what you've learned in Hour 8, do the following problems. Check your answers with the solution key in Appendix B.

1. Solve for x: $3(x - 4) - 4(x + 2) = 2(2x - 5)$
a. $x = -\frac{6}{5}$
b. $x = -1$
c. $x = -2$
d. $x = -3$

2. Five times the greater of two consecutive integers is 4 less than six times the smaller integer. What are the integers?
a. 8 and 9
b. 9 and 10
c. 10 and 11
d. 6 and 7

3. Simplify: $4a(3a^2 + 5a - 2) =$
a. $12a^3 + 9a^2 - 8a$
b. $12a^3 + 20a^2 - 6a$
c. $12a^3 + 9a^2 + 8a$
d. $12a^3 + 20a^2 - 8a$

Review

4. Simplify: $-3x^3(2x^2 - 5x - 1) =$
a. $-6x^5 + 15x^4 + 3x^3$
b. $6x^5 - 15x^4 + 3x^3$
c. $-6x^5 + 15x^4 - 3x^3$
d. $-6x^5 - 15x^4 - 3x^3$

5. Simplify: $x^3(2x - 5) + x(6x^3 + x^2) =$
a. $-4x^4 - 4x^3$
b. $-8x^4 - 4x^3$
c. $8x^4 - 4x^3$
d. $-8x^4 + 4x^3$

6. Simplify: $x^3(2x - 5) - x(6x^3 + x^2) =$
a. $-4x^4 - 6x^3$
b. $-4x^4 - 4x^3$
c. $2x^4 - 11x^3 + x^2$
d. $-6x^4 - 4x^3$

7. Simplify: $(x + 3)(x^2 - 2x + 5) =$
a. $x^3 + x^2 + 11x + 15$
b. $x^3 + x^2 - x + 15$
c. $x^3 - x^2 - x - 15$
d. $x^3 - x^2 + 11x + 15$

Review

Use the FOIL method to simplify problems 8 through 10.

8. Simplify: $(x + 3)(x + 4) =$
a. $x^2 + 7x + 12$
b. $x^2 + 12x + 7$
c. $x^2 + 7x + 7$
d. $x^2 + 12x + 12$

9. Simplify: $(x + 5)(x + 2) =$
a. $x^2 + 10x + 10$
b. $x^2 + 7x + 10$
c. $x^2 + 7x + 7$
d. $x^2 + 10x + 7$

10. Simplify: $(x - 7)(x + 7) =$
a. $x^2 + 7x - 49$
b. $x^2 - 7x + 49$
c. $x^2 - 49$
d. $x^2 + 49$

Hour 9

Polynomials: Dividing

CHAPTER SUMMARY

In this hour you will learn about …

- Dividing monomials
- Dividing polynomials by monomials

In the past few hours, you have progressed through adding, subtracting, and multiplying polynomial expressions, seeing how the work you did in the first hours of learning algebra helped you solve rather complicated polynomial problems. As you learn how to divide polynomials this hour, remember that division is closely related to multiplication. Because you have already mastered multiplying, dividing polynomials will come even more easily.

Dividing Monomials

When monomials are divided, as in "12b divided by 3b," the problem can be written in two different ways:

With the division symbol, such as $12b \div 3b$

or

In fraction form, such as $\frac{12b}{3b}$.

To simplify $12b \div 3b$, the fraction form is usually easier to work with because it puts the monomials over each other and helps you keep the terms in a convenient order. As the polynomials get more complex, it will also be easier to do the division problems in fraction form.

JUST A MINUTE

When dividing variables, as in monomials, the sign rules are the same as for numbers: dividing two positive terms gives a positive answer, dividing two negative terms gives a positive answer, and dividing two terms with different signs gives a negative answer.

Let's simplify the division problem that has been the example so far. First, separate it into two multiplied fractions, one fraction for the coefficients and one fraction for the variables:

$$\frac{12b}{3b} = \frac{12}{3} \cdot \frac{b}{b}$$

Now you have two simple problems that, when multiplied, will give you the final answer:

$$\frac{12}{3} = 4 \text{ and } \frac{b}{b} = 1 \text{ (any number divided by itself always equals 1)}$$

Multiplying the answer to each fraction, you get $4 \cdot 1 = 4$. The simplified answer for $\frac{12b}{3b}$ is 4.

That problem was simple, but this process will also work for more complex monomials that include exponents, such as the following:

$$\frac{x^4y^3}{xy^2}$$

As in the last example, separate the problem into two fractions, this time making one fraction for the *x* variables and another one for the *y* variables:

$$\frac{x^4}{x} \cdot \frac{y^3}{y^2}$$

This problem is more complex than the previous one, due to the exponents in some of the factors. Think back to the multiplication rules for factors that include exponents. As you recall, when multiplying factors that have the same base, you simply keep the base and add the exponents. But in this problem, the *x* variables are being *divided*, as are the *y* variables. So you need a plan for dividing monomials that have the same base.

Let's look at just the first fraction in the problem and take it one step at a time:

$$\frac{x^4}{x}$$

GO TO ◄
Refer to Hour 7 to review multiplying monomials.

Written in expanded form, it would become:

$$\frac{x^4}{x} = \frac{x \cdot x \cdot x \cdot x}{x}$$

Separating it into two fractions gives the following. (Notice that the second fraction's denominator is 1, from the identity property. This keeps both parts of the solution in fraction form, which is helpful as you work toward the final answer.)

$$\frac{x^4}{x} = \frac{x \cdot x \cdot x \cdot x}{x} = \frac{x}{x} \cdot \frac{x \cdot x \cdot x}{1}$$

JUST A MINUTE

The identity property states that any number or term multiplied by 1 is equal to itself. $3 \cdot 1 = 3$; $x \cdot 1 = x$

Any term divided by itself equals 1, so $\frac{x}{x} = 1$. This leads to the following final answer:

$$\frac{x^4}{x} = \frac{x \cdot x \cdot x \cdot x}{x} = \frac{x}{x} \cdot \frac{x \cdot x \cdot x}{1} = 1 \cdot \frac{x \cdot x \cdot x}{1} = x \cdot x \cdot x = x^3$$

This step-by-step process shows that when you begin with $\frac{x^4}{x}$, the simplified answer is x^3. Now look at the expanded form of the second part of the original problem to see its simplified answer:

$$\frac{y^3}{y^2} = \frac{y \cdot y \cdot y}{y \cdot y} = \frac{y}{y} \cdot \frac{y}{y} \cdot \frac{y}{1} = 1 \cdot 1 \cdot \frac{y}{1} = y$$

This step-by-step process shows that when you begin with $\frac{y^3}{y^2}$, the simplified answer is y.

You probably saw the rule developing as you worked through the previous examples. The division rule for exponents says that, when dividing variables with the same base, you subtract the exponents. This rule saves a great amount of time and trouble, and you use it every time a problem includes dividing monomial terms, as shown in the following problems.

Divide the following monomials:

a. $\dfrac{14x^4y^3}{2x} = \dfrac{14}{2} \cdot \dfrac{x^4}{x} \cdot \dfrac{y^3}{1} = 7x^3y^3$

b. $\dfrac{-9ab^5}{3ab} = \dfrac{-9}{3} \cdot \dfrac{a}{a} \cdot \dfrac{b^5}{b} = -3b^4$

c. $\dfrac{20x^3y^6}{8xy^2} = \dfrac{20}{8} \cdot \dfrac{x^3}{x} \cdot \dfrac{y^6}{y^2} = \dfrac{4 \cdot 5}{4 \cdot 2} \cdot x^2y^4 = \dfrac{5}{2}x^2y^4$

Look again at examples a and b. The division rule for exponents (when dividing the same base, keep the base and subtract the exponents) allows you to go directly from the problem to the answer, especially if the coefficients are easily divided, as in these problems. The coefficients in example c are slightly more complex, but simplifying the variables without the "work" step should be possible.

In all the examples so far, the exponents in the *numerator* were larger than the exponents in the *denominator*. Let's look at an example that is different:

STRICTLY DEFINED

The upper part of a fraction is called the **numerator;** the lower part of a fraction is called the **denominator.**

$$\frac{a^3}{a^5}$$

Write this problem in expanded form to see the detailed process:

$$\frac{a^3}{a^5} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{1}{a \cdot a} = 1 \cdot 1 \cdot 1 \cdot \frac{1}{a \cdot a} = \frac{1}{a^2}$$

This step-by-step process shows that when you begin with $\dfrac{a^3}{a^5}$, the simplified answer is $\dfrac{1}{a^2}$.

But don't forget that when you are dividing the same base, you keep the base and subtract the exponents. In the example just completed, subtracting the exponents 3 and 5 gives –2, and the final answer in that problem was $\dfrac{1}{a^2}$. When you divide monomial terms and get a negative exponent, you can make the exponent positive by moving the base with its exponent to the denominator—for example, $x^{-3} = \dfrac{1}{x^3}$ and $b^{-5} = \dfrac{1}{b^5}$.

The following example reviews the processes shown in the previous problems:

$$\frac{-18x^3yz^2}{4x^7yz} = \frac{-18}{4} \cdot \frac{x^3}{x^7} \cdot \frac{y}{y} \cdot \frac{z^2}{z} = \frac{2}{2} \cdot \frac{-9}{2} \cdot \frac{1}{x^4} \cdot \frac{z}{1} = \frac{-9z}{2x^4}$$

You should be able to leave out some of the expansion steps in this type of division problem, but you are free to decide how much of the process to write down and how much to do mentally.

Now try these monomial division problems, checking your answers on page 106:

1. $\frac{20x^6y^2}{2x} =$
2. $\frac{-15a^4b}{3ab} =$
3. $\frac{21x^4y^5z^4}{4x^6y^2z} =$

The skills needed for dividing one monomial by another allow you to also divide a polynomial by a monomial using almost the same process. The problems you will see as you finish this hour include numerators with two or three terms—binomials and trinomials. The goal is to simplify the problems into lowest terms, using the exponent and fraction rules as you did in the monomial division problems.

TIME SAVER

Division can be shown with the ÷ symbol or as a fraction. The fraction form is easier to work with when polynomials are in the problem.

Consider the following division problem:

$$(8x^4 + 6x^3) \div 2x^2$$

The most efficient way to divide is in fraction form, so this problem is best written as:

$$\frac{8x^4 + 6x^3}{2x^2}$$ (The parentheses are not needed in fraction form.)

In fractions, a numerator with two added terms can be written as the sum of two fractions. This changes the problem into two monomial division problems, as follows:

$$\frac{8x^4 + 6x^3}{2x^2} = \frac{8x^4}{2x^2} + \frac{6x^3}{2x^2}$$

Simplify each of these fractions:

$$\frac{8x^4 + 6x^3}{2x^2} = \frac{8x^4}{2x^2} + \frac{6x^3}{2x^2} = 4x^2 + 3x$$

$4x^2 + 3x$ cannot be combined because the x variables don't have the same exponent—they're not like terms—so this is the final answer. The coefficients were easily divided, making this problem easy to do mentally, without the expansion steps.

JUST A MINUTE

Terms with different exponents are not like terms, so they cannot be added or subtracted.

If the numerator is a trinomial, separate the division problem into three fractions, as in the following example (note that the subtraction sign in the numerator becomes a subtraction sign in front of the third fraction):

$$\frac{10a^3 + 2a^4 - 24a^5}{2a^3} = \frac{10a^3}{2a^3} + \frac{2a^4}{2a^3} - \frac{24a^5}{2a^3} = 5 + a - 12a^2$$

The answer can also be written as $-12a^2 + a + 5$, with the terms arranged in descending order by their exponents. Either is correct, but descending order is often the preferred order.

Now try these division problems, checking your answers on page 106.

4. $\dfrac{16x^6 + 20x^4}{4x^2} =$

5. $\dfrac{9a^4 - 12a^8 + 3a^{10}}{3a^2} =$

6. $\dfrac{6x^8 - 5x^4 - 3x^2}{2x^2} =$

Let's review how to combine polynomials that include adding, subtracting, and multiplying, because all of these operations are important in learning how to factor polynomials in Hour 10.

Follow the steps in the examples that follow, focusing in each problem on which operation is used and being careful about the exponents.

Remember that in addition and subtraction, the exponents won't change, but in multiplying they do.

JUST A MINUTE

Addition and subtraction require like terms: the same variable and exponents. Subtraction can be changed to addition if you change the signs of the terms after the subtraction sign. The order of operations rule requires that you multiply and divide first, then add and subtract.

Simplify:

$$9a^2b + 2b^3 - 12a^2b - 7 + 6b^3 =$$
$$9a^2b - 12a^2b + 2b^3 + 6b^3 - 7 =$$
$$-3a^2b + 8b^3 - 7$$

In this example, the like terms are moved next to each other in step two and combined to make the final answer. As you are more comfortable with this type of problem, you can easily skip step two and go straight to the answer.

Simplify:

$$(3x^4 - 7x^2 + 6) + (10x^4 + 5x^2 + 1) =$$
$$3x^4 + 10x^4 - 7x^2 + 5x^2 + 6 + 1 =$$
$$13x^4 - 2x^2 + 7$$

In this addition problem, the like terms in the two trinomials are moved next to each other, as in the prior example. Again, you may skip this step, if you prefer.

Simplify:

$$(-3a^3 + 6a^2 - 9a + 1) - (a^3 + 5a^2 - 6a) =$$
$$-3a^3 + (-a^3) + 6a^2 + (-5a^2) - 9a + (+6a) + 1 =$$
$$-4a^3 + a^2 - 3a + 1$$

This subtraction problem is more complex, so it is wise to change the signs of the terms in the second trinomial and then move the like terms together before writing the final answer. You will avoid making errors that can occur when you try to do too many steps at the same time.

Simplify:

$(5x^2y)(2xy^3) + (3y^2)(x^3y^2) =$
$10x^3y^4 + 3x^3y^4 =$
$13x^3y^4$

The order of operations requires you to do the multiplying first and then the adding. This problem needs the work shown in step two because two operations are involved.

Now try these polynomial review problems, checking your answers below.

7. $x^4 - 8x^3 + 2x^4 + 4x^3 + 7 =$
8. $(6x^3 + 5x^2 + x - 2) - (3x^3 - x^2 + 6x + 2) =$
9. $(2ab^2)(6a^2b^2) + (b^3)(3a^3b) =$

Answers to Sample Problems

1. $\dfrac{20x^6y^2}{2x} = \dfrac{20}{2} \cdot \dfrac{x^6}{x} \cdot \dfrac{y^2}{1} = 10x^5y^2$

2. $\dfrac{-15a^4b}{3ab} = \dfrac{-15}{3} \cdot \dfrac{a^4}{a} \cdot \dfrac{b}{b} = -5a^3$

3. $\dfrac{21x^4y^5z^4}{4x^6y^2z} = \dfrac{21}{4} \cdot \dfrac{x^4}{x^6} \cdot \dfrac{y^5}{y^2} \cdot \dfrac{z^4}{z} = \dfrac{21y^3z^3}{4x^2}$

4. $\dfrac{16x^6 + 20x^4}{4x^2} = \dfrac{16x^6}{4x^2} + \dfrac{20x^4}{4x^2} = 4x^4 + 5x^2$

5. $\dfrac{9a^4 - 12a^8 + 3a^{10}}{3a^2} = \dfrac{9a^4}{3a^2} - \dfrac{12a^8}{3a^2} + \dfrac{3a^{10}}{3a^2} = 3a^2 - 4a^6 + a^8$

6. $\dfrac{6x^8 - 5x^4 - 3x^2}{2x^2} = \dfrac{6x^8}{2x^2} - \dfrac{5x^4}{2x^2} - \dfrac{3x^2}{2x^2} = 3x^6 - \dfrac{5x^2}{2} - \dfrac{3}{2}$ or

   $3x^6 - \dfrac{5}{2}x^2 - \dfrac{3}{2}$

7. $x^4 - 8x^3 + 2x^4 + 4x^3 + 7 = 3x^4 - 4x^3 + 7$
8. $(6x^3 + 5x^2 + x - 2) - (3x^3 - x^2 + 6x + 2) =$
   $6x^3 + (-3x^3) + 5x^2 + (+x^2) + x + (-6x) - 2 + (-2) =$
   $3x^3 + 6x^2 - 5x - 4$
9. $(2ab^2)(6a^2b^2) + (b^3)(3a^3b) = 12a^3b^4 + 3a^3b^4 = 15a^3b^4$

Hour's Up!

Based on what you've learned in Hour 9, do the following division problems. Check your answers with the solution key in Appendix B.

1. $\frac{15x^8y^4}{3x} =$

a. $12x^7y^4$
b. $5x^7y^4$
c. $5x^8y^4$
d. $12x^8y^4$

2. $\frac{-24a^3b^5}{3ab^5} =$

a. $-8a^2$
b. $8a^2$
c. $-8a^2b$
d. $8a^2b$

3. $\frac{18x^2y^4z^6}{-8x^5y^3z} =$

a. $\frac{-9yz^6}{4x^3}$
b. $\frac{-10yz^5}{x^3}$
c. $\frac{-9z^5}{4x^3y}$
d. $\frac{-9yz^5}{4x^3}$

4. $\frac{2xy^3z^2}{9x^4y^2z^5} =$

a. $\frac{2y^2}{9x^3z^3}$
b. $\frac{2y}{9x^2z^3}$
c. $\frac{2y}{9x^3z}$
d. $\frac{2y}{9x^3z^3}$

Review

5. $\dfrac{20x^8 + 16x^6}{2x^4} =$

a. $18x^4 + 14x^2$
b. $10x^4 + 8x^2$
c. $18x^2 + 8x^4$
d. $10x^2 + 8x^4$

6. $\dfrac{3a^6 - 21a^4 + 15a^2}{3a^2} =$

a. $a^3 - 7a^2 + 5$
b. $a^3 - 7a^2 + 5$a
c. $a^4 - 7a^2 + 5$
d. $a^4 - 18a^4 + 12$

7. $\dfrac{15x^{10} - 35x^8 - 5x^4}{5x^4} =$

a. $3x^6 - 7x^4 - 1$
b. $12x^6 - 30x^4$
c. $3x^6 - 7x^2 - 1$
d. $12x^6 - 30x^4 - 1$

8. $\dfrac{6a^6 - 2a^4 + 4a^3}{3a^3} =$

a. $3a^3 -)\dfrac{2}{3}a + \dfrac{4}{3}$
b. $2a^3 - \dfrac{2}{3}a + \dfrac{4}{3}$
c. $2a^3 - 2a + 4$
d. $2a^2 - \dfrac{2}{3}a + \dfrac{4}{3}$

9. $\dfrac{2x^4y^2 + 6x^2y^4}{2x^2y^2} =$

a. $x^2y + 3y^2$
b. $x^2y^2 + 3x^2y^2$
c. $x^2 + 4y^2$
d. $x^2 + 3y^2$

10. $\dfrac{-8x^4 + 10x^3 - 12x^2}{2x^2} =$

a. $-6x^2 + 8x - 10$
b. $-4x^2 + 8x - 6$
c. $-4x^2 + 5x - 6$
d. $-6x^2 + 5x - 6$

Review

Hour 10

Polynomials: Solving Equations and Factoring

CHAPTER SUMMARY

In this hour you will learn about …

- Solving factored equations
- Factoring trinomials
- Solving equations by factoring

Now that you are able to simplify polynomial expressions with the four operations, it's time to move toward solving equations that include more complex polynomial terms. The equations you learned how to solve in Hours 5 and 6 had variables that were simple—they had exponents of 1. By the end of this hour, you'll be able to solve equations like $x^2 + 6x + 8 = 0$ or $x^2 - 7x + 12 = 0$.

As with almost every hour of studying algebra, learning to solve these equations evolves as you build on prior knowledge and adapt it to meet your goal. Combining binomial multiplication from Hour 8 with the basic equations from Hour 5, you'll know how to solve higher-level equations.

Solving Factored Equations

When you learned the multiplication tables, you were really working with *factors*. Because 5 times 6 equals 30, 5 and 6 are called factors of 30. Also, 3 and 10 are factors of 30, as are 2 and 15.

STRICTLY DEFINED

Factors are numbers or variables that are multiplied to give a product. 2 and 5 are factors of 10. 1 and 10 are factors of 10. 1, 2, and 5 are factors of 10. 1 is a factor of all numbers because of the identity property of multiplication: $1 \cdot x = x$. Every number is a factor of itself: 10 is a factor of 10.

From your work in algebra, you know that 3 times x equals $3x$, so 3 and x are called factors of $3x$. Because c times d equals cd, c and d are factors of cd. At its simplest, the factors of an expression are the numbers or variables that can be multiplied to create the expression.

During Hour 8, you learned how to multiply polynomials, so you were also learning about factors. You used the FOIL process to multiply the binomials $(x + 2)$ and $(x + 4)$:

$(x + 2) \cdot (x + 4) = x^2 + 4x + 2x + 8 = x^2 + 6x + 8$

Multiplying $(x + 2)$ times $(x + 4)$ equals $x^2 + 6x + 8$, so $(x + 2)$ and $(x + 4)$ are factors of $x^2 + 6x + 8$.

Similarly, multiplying $(x - 6)$ times $(x + 4)$ with the FOIL method gives:

$(x - 6) \cdot (x + 4) = x^2 + 4x - 6x - 24 = x^2 - 2x - 24$

Therefore, $(x - 6)$ and $(x + 4)$ are factors of $x^2 - 2x - 24$.

You may be wondering why it matters that these binomials are factors of the trinomials. It's all about the goal for this hour: to solve an equation like $x^2 + 6x + 8 = 0$. You can easily solve equations like $x + 6 = 8$ or even $3x + 7 = 13$. It's the x^2 in that trinomial that is causing the trouble. By following the solution steps of the next sample equation, you will see how factors can be used to solve an equation, and that will prepare you for solving $x^2 + 6x + 8 = 0$.

The following example is a simpler equation than the trinomial one, but it shows an important principle that leads you toward the more complex equations, one step at a time.

Consider: $x \cdot y = 0$

In words, you could write this as "x times y equals 0." You know that factors are numbers or variables that multiply to give a product. Because x and y are being multiplied to make 0, x and y are factors of 0 in this problem.

Now recall from the basic multiplication tables that any number times 0 always equals 0. So in "x times y equals 0," there must be two solutions: either x is 0 or y is 0.

To verify these solutions, put 0 in for x in the equation $x \cdot y = 0$, and you get $0 \cdot y = 0$, which is true. And if you put 0 in for y in the same equation $x \cdot y = 0$, you get $x \cdot 0 = 0$, which is also true.

Therefore, the solutions to $x \cdot y = 0$ are $x = 0$ or $y = 0$ because either of these values will make a true equation.

The logical extension of the previous example is that if any two factors are multiplied to make zero, then either one of the factors is equal to zero or the other factor is equal to zero. This is also true for binomials that are multiplied in an equation, as in the following example.

Solve: $(x + 2) \cdot (x + 4) = 0$

JUST A MINUTE

Multiplication can be indicated by a · or by parentheses with no operation shown. For example, $(x + 1)(x - 3)$ is the same multiplication problem as $(x + 1) \cdot (x - 3)$. Both forms are common in algebraic multiplication and have the same meaning.

Each binomial is a factor of 0 in this equation. So each binomial can equal 0. You write this in algebra as:

$x + 2 = 0$ or $x + 4 = 0$

The good news is that you can easily solve each of these simple equations:

$x + 2 = 0$	*or*	$x + 4 = 0$
$x + 2 - 2 = 0 - 2$		$x + 4 - 4 = 0 - 4$
$x = -2$		$x = -4$

To check, put –2 in for x in the original equation:

$$(-2 + 2) \cdot (-2 + 4) = 0 \rightarrow (0) \cdot (2) = 0 \rightarrow 0 = 0;$$

Therefore, $x = -2$ is a solution.

Also put –4 in for x in the original equation:

$$(-4 + 2) \cdot (-4 + 4) = 0 \rightarrow (-2) \cdot (0) = 0 \rightarrow 0 = 0;$$

Therefore, $x = -4$ is a solution.

Let's restate the equation and show the work that leads directly to its solutions:

Solve: $(x + 2) \cdot (x + 4) = 0$

Solution: $x + 2 = 0$ or $x + 4 = 0$

$x + 2 - 2 = 0 - 2$ $\quad$ $x + 4 - 4 = 0 - 4$
$x = -2$ $\quad$ $x = -4$

–2 and –4 are the solutions because $(-2 + 2)(-2 + 4) = 0 \rightarrow 0 \cdot 2 = 0$ and $(-4 + 2)(-4 + 4) = 0 \rightarrow -2 \cdot 0 = 0$.

This process works for all binomials that are multiplied to equal zero. Follow the solution steps in the next two samples.

Solve: $(a - 8)(a + 5) = 0$

Solution: $a - 8 = 0$ *or* $a + 5 = 0$

$a - 8 + 8 = 0 + 8$ $\quad$ $a + 5 - 5 = 0 - 5$
$a = 8$ $\quad$ $a = -5$

GO TO ▶

To solve an equation using the binomial factor method, the equation must be written with the factors on one side of the equals sign and 0 on the other side. For example, $(x + 5) \cdot (x - 3) = 0$ can be solved with the factor method; $(x + 5) \cdot (x - 3) = 6$ cannot be solved with the factor method. Go to Hour 24 to learn how to solve the second equation.

8 and –5 are the solutions because $(8 - 8)(8 + 5) = 0 \rightarrow (0)(13) = 0$ and also $(-5 - 8)(-5 + 5) = 0 \rightarrow (-13)(0) = 0$.

Equations with more complex binomials can also be solved with this method, as shown in the following example.

Solve: $(3x + 5)(x - 2) = 0$

Solution: $3x + 5 = 0$ *or* $x - 2 = 0$

$3x + 5 - 5 = 0 - 5$ $\quad$ $x - 2 + 2 = 0 + 2$
$3x = -5$ $\quad$ $x = 2$
$3x \div 3 = -5 \div 3$
$x = -\frac{5}{3}$

$-\frac{5}{3}$ and 2 are the solutions because $(3 \cdot -\frac{5}{3} + 5)(-\frac{5}{3} - 2) = 0 \rightarrow (-5 + 5)(-\frac{5}{3} - 2) = 0 \rightarrow (0)(-\frac{5}{3} - 2) = 0 \rightarrow 0 = 0$ and also $(3 \cdot 2 + 5)(2 - 2) = 0 \rightarrow (6 + 5)(0) = 0 \rightarrow (11)(0) = 0 \rightarrow 0 = 0$.

Although they are not often seen in basic algebra, equations that have three or more binomial factors with a product of zero can be solved with the same method, as shown in the following example.

Solve: $(x - 4)(x + 2)(x + 5) = 0$

Solution: With three binomial factors, there will be three solutions to this equation because if any one of the binomials equals zero, the entire product will also equal zero. So just set each of the binomial factors equal to zero and solve the simple equations.

$x - 4 = 0$	$x + 2 = 0$	$x + 5 = 0$
$x - 4 + 4 = 0 + 4$	$x + 2 - 2 = 0 - 2$	$x + 5 - 5 = 0 - 5$
$x = 4$	$x = -2$	$x = -5$

To check, put 4 in for x:

$$(4 - 4)(4 + 2)(4 + 5) = 0 \rightarrow (0)(6)(9) = 0 \rightarrow 0 = 0$$

Put –2 in for x:

$$(-2 - 4)(-2 + 2)(-2 + 5) = 0 \rightarrow (-6)(0)(3) = 0 \rightarrow 0 = 0$$

Put –5 in for x:

$$(-5 - 4)(-5 + 2)(-5 + 5) = 0 \rightarrow (-9)(-3)(0) = 0 \rightarrow 0 = 0$$

Therefore, 4, –2, and –5 are the solutions.

Now try solving these equations, checking your answers on page 120.

1. $(x + 8)(x + 3) = 0$
2. $(c - 5)(c + 1) = 0$
3. $(2a - 3)(a - 6) = 0$

Solving equations that have factors on one side and zero on the other is relatively simple and prepares you for meeting your goal of solving equations like $x^2 + 6x + 8 = 0$. You just need to be able to change the trinomial in this equation into factors, and then you can solve it easily.

Factoring Trinomials

Earlier this hour, you reviewed the FOIL method of multiplying binomials to create trinomials. Let's look at another example of the FOIL process.

Multiply: $(x + 5)(x + 4)$

Solution: $(x + 5)(x + 4) = x^2 + 4x + 5x + 20 = x^2 + 9x + 20$

JUST A MINUTE

The FOIL method allows you to multiply two binomials very quickly. F, O, I, and L represent the terms in the binomials and remind you what terms to multiply: F is for "first terms," O is for "outside terms," I is for "inside terms," and L is for "last terms." For example, $(x + 3)(x + 6) = x \cdot x + 6 \cdot x + 3 \cdot x + 3 \cdot 6 = x \cdot x + 9x + 18$.

Look at the same problem in reverse order. The trinomial first, the middle step, and then the binomial factors would look like this:

$$x^2 + 9x + 20 = x^2 + 4x + 5x + 20 = (x + 5)(x + 4)$$

If you look at the trinomial— $x^2 + 9x + 20$—and compare it to the binomial factors—$(x + 5)(x + 4)$—you should be able to see from where each of the three trinomial terms—x^2, $9x$, and 20—come. You are basically "undoing" the FOIL process to find the binomial factors.

Factoring trinomials simply means finding the binomial terms that were multiplied to create the trinomial. Looking at each of the trinomial terms gives you a clue for finding the binomial factors. Using words to explain this process is much more difficult than just doing it, so review the example that follows to see how to factor a trinomial.

Here's the step-by-step process, based on the same problem:

$$x^2 + 9x + 20 = x^2 + 4x + 5x + 20 = (x + 5)(x + 4)$$

1. x^2 came from the x terms in the binomials. (first terms)
2. $9x$ came from adding $4x$ and $5x$. (outside and inside terms)
3. 20 came from multiplying 4 times 5. (last terms)

JUST A MINUTE

The last terms in the binomials must sum to give the coefficient of the middle trinomial term, and they must multiply to give the third trinomial term.

Now look at a new trinomial and see the steps for factoring it:

$$x^2 + 7x + 10 = x^2 + 2x + 5x + 10 = (x + 5)(x + 2)$$

1. x^2 came from the x terms in the binomials. (first terms)
2. $7x$ came from adding $2x$ and $5x$. (outside and inside terms)
3. 10 came from multiplying 2 times 5. (last terms)

When you can see the middle step in the previous examples, it's easy to know where the outside and inside terms come from. But when you are factoring trinomials, you won't have the answer to help you, so you need a plan. Check out the previous Just a Minute note and the Time Saver note below for helpful hints.

It's all about comparing the trinomial's middle term and last term to find the correct numbers to put in the binomials. With practice, you'll be able to go straight from the trinomial to the binomial factors—but it will take some practice.

The following examples help you get started. To factor each trinomial, use the three steps from the prior examples as a guide.

a. Factor $x^2 + 8x + 15$.

You are looking for two numbers that sum to 8 and multiply to give 15. The number pairs that multiply to 15 are 1 and 15 and also 3 and 5. Do any of these pairs sum to 8? Yes: 3 and 5. Use them in the binomials:

Solution: $x^2 + 8x + 15 = (x + 3)(x + 5)$

FOIL the binomials to check: $(x + 3)(x + 5) = x^2 + 5x + 3x + 15 = x^2 + 8x + 15$; therefore, $(x + 3)$ and $(x + 5)$ are the factors of $x^2 + 8x + 15$.

TIME SAVER

When factoring trinomials, focus on the third term and find all the number pairs that multiply to give that term. Then choose the pair that sums to the coefficient of the trinomial's middle term.

b. Factor $x^2 + 7x + 12$.

Find a pair of numbers that sum to 7 and multiply to give 12. The number pairs that multiply to give 12 are 1 and 12, 2 and 6, 3 and 4. Do any of these sum to 7? Yes, 3 and 4 will work. Use them in the binomials:

Solution: $x^2 + 7x + 12 = (x + 3)(x + 4)$

FOIL the binomials to check: $(x + 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$; therefore, $(x + 3)$ and $(x + 4)$ are the factors of $x^2 + 7x + 12$.

c. Factor $x^2 - 7x + 12$.

Even with a $-7x$ in this trinomial, the process is the same—just watch the signs. You know that $3 \cdot 4$ equals 12, but you need the number pair to sum to –7. So use $-3 \cdot -4$. This gives +12 when multiplied and -7 when added. Perfect!

Solution: $x^2 - 7x + 12 = (x - 3)(x - 4)$

FOIL the binomials to check: $(x - 3)(x - 4) = x^2 - 4x - 3x + 12 = x^2 - 7x + 12$; therefore, $(x - 3)$ and $(x - 4)$ are the factors of $x^2 - 7x + 12$.

d. Factor $x^2 + 3x - 10$.

This trinomial is a little more confusing, with the last term being a –10. Remember that multiplying a positive number times a negative number always gives a negative answer. So you need to find a number pair that multiplies to give a –10 and sums to give a +3. Focus on the numbers that multiply to give 10: 1 and 10, 2 and 5. One of the numbers in the pair will need to be negative to give a product of –10. Which pair will sum to +3? –2 added to 5 will give +3, and multiplying –2 times 5 will give –10, so use –2 and 5.

Solution: $x^2 + 3x - 10 = (x - 2)(x + 5)$

FOIL the binomials to check: $(x - 2)(x + 5) = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$; therefore, $(x - 2)$ and $(x + 5)$ are the factors of $x^2 + 3x - 10$.

TIME SAVER

The key to knowing the signs of the last terms in the binomial factors is to check the sign of the third trinomial term: a negative third term tells you the signs must be a + and a –. A positive third term tells you the signs must be either both + or both –. This helps you narrow down the possible signs for the last terms in the binomials.

e. Factor $x^2 - 8x + 15$.

Remember that multiplying two negative numbers will always give a positive answer, and adding two negatives will always give a negative answer. So you need a pair of negative numbers that multiplies to 15 and sums to –8. The pairs that multiply to 15 are –1 and –15, –3 and –5. Which pair sums to –8? The –3 and –5. Use them in the binomials.

Solution: $x^2 - 8x + 15 = (x - 3)(x - 5)$

FOIL the binomials to check: $(x - 3)(x - 5) = x^2 - 5x - 3x + 15 = x^2 - 8x + 15$; therefore, $(x - 3)$ and $(x - 5)$ are the factors of $x^2 - 8x + 15$.

A special case that often comes up is factoring a binomial such as $x^2 - 16$.

Recall from the FOIL examples in Hour 8 that if the last terms in the binomials are opposites of each other, you will get an answer that isn't a

trinomial, because the middle terms sum to 0. Factoring $x^2 - 16$ is actually easier than factoring a trinomial, as you see from the solution that follows.

Factor $x^2 - 16 = (x - 4)(x + 4)$

TIME SAVER

Finding the binomial factors for the difference of two squared terms is actually the easiest of all factoring, since the signs of the last terms will always be + and – —in other words, the last terms will be opposites.

FOIL the binomials to check: $(x - 4)(x + 4) = x^2 + 4x - 4x - 16 = x^2 - 16$; therefore, $(x - 4)$ and $(x + 4)$ are the factors of $x^2 - 16$.

This trinomial factoring process brings together many of the algebra skills you have learned so far. By watching the signs of the terms and finding the possible number pairs that multiply to give the trinomial's last term, you can find the binomial factors. Be aware that some trinomials cannot be factored; if they are part of an equation you are trying to solve, you will learn in Hour 24 how to isolate their variable. For now, all the trinomials in this hour will factor into two binomials if you follow the process shown in the prior examples.

Now try to factor these polynomials into two binomials, checking your answers on page 120.

4. Factor $x^2 + 9x + 20$.
5. Factor $x^2 - 8x + 12$.
6. Factor $x^2 - 6x - 16$.
7. Factor $x^2 - 25$.

All this factoring work has prepared you to reach this hour's goal, to solve equations like $x^2 + 6x + 8 = 0$ or $x^2 - 7x + 12 = 0$. Finding the trinomial's factors is the key.

Solving Equations by Factoring

If an equation has a trinomial that is equal to zero, and if that trinomial can be factored, you can solve the equation. Recall the problems you worked on earlier this hour, such as the following:

Solve $(x - 5) \cdot (x + 2) = 0$

JUST A MINUTE

Recall that if two or more factors multiply to give zero, each of the factors could equal zero. Let each of the factors equal zero, and then isolate the variable.

Since each binomial is a factor of 0, either $(x - 5)$ equals 0 or $(x + 2)$ equals 0. Write and solve these two equations:

$$\begin{array}{lll} x - 5 = 0 & \textit{or} & x + 2 = 0 \\ x - 5 + 5 = 0 + 5 & & x + 2 - 2 = 0 - 2 \\ x = 5 & & x = -2 \end{array}$$

Now consider solving the equation $x^2 - 3x - 10 = 0$.

If you can factor the trinomial, you will be able to solve the equation.

Let's factor $x^2 - 3x - 10$. What two numbers multiply to give –10 and sum to –3? The number pair –5 and 2 will do this, so use them in the binomials.

Since $x^2 - 3x - 10 = (x - 5)(x + 2)$, the equation can be written as:

$$(x - 5) \cdot (x + 2) = 0$$

You solved this equation earlier, using the factors of 0 process, so you know the solutions are 5 and –2. This example shows that you simply need to factor a trinomial equation to find its solutions.

The following examples show you the step-by-step process.

PROCEED WITH CAUTION

Be sure to check each of the answers to verify that they are correct. Each solution needs to be substituted for the variable in the squared term and in the nonsquared term, and this checking process needs to be done twice—once for each solution.

f. Solve: $x^2 + 8x + 12 = 0$

Solution: Factor the trinomial and solve the two equations.

$$\begin{array}{lll} x^2 + 8x + 12 = 0 & & \\ (x + 6)(x + 2) = 0 & & \\ x + 6 = 0 & \textit{or} & x + 2 = 0 \\ x + 6 - 6 = 0 - 6 & & x + 2 - 2 = 0 - \\ x = -6 & \textit{or} & x = -2 \end{array}$$

To check, put –6 in the original equation for *x:*

$$(-6)^2 + 8 \cdot -6 + 12 = 0 \rightarrow 36 - 48 + 12 = 0 \rightarrow -12 + 12 = 0 \rightarrow 0 = 0$$

Put –2 in the original equation for *x:*

$$(-2)^2 + 8 \cdot -2 + 12 = 0 \rightarrow 4 - 16 + 12 = 0 \rightarrow -12 + 12 = 0 \rightarrow 0 = 0$$

Therefore, –6 and –2 are the solutions to $x^2 + 8x + 12 = 0$.

The next two examples show how to solve the equations that were set as the goal as you began this hour:

g. Solve: $x^2 + 6x + 8 = 0$

Solution: Factor the trinomial and solve the two equations.

$$x^2 + 6x + 8 = 0$$
$$(x + 2)(x + 4) = 0$$
$$x + 2 = 0 \quad \textit{or} \quad x + 4 = 0$$
$$x + 2 - 2 = 0 - 2 \qquad x + 4 - 4 = 0 - 4$$
$$x = -2 \qquad\qquad x = -4$$

To check, put –2 in the original equation for *x:*

$$(-2)^2 + 6 \cdot -2 + 8 = 0 \rightarrow 4 - 12 + 8 = 0 \rightarrow -8 + 8 = 0 \rightarrow 0 = 0$$

Put –4 in the original equation for *x:*

$$(-4)^2 + 6 \cdot -4 + 8 = 0 \rightarrow 16 - 24 + 8 = 0 \rightarrow -8 + 8 = 0 \rightarrow 0 = 0$$

Therefore, –2 and –4 are solutions to $x^2 + 6x + 8 = 0$.

h. Solve: $x^2 - 7x + 12 = 0$

TIME SAVER

It is possible to have only one solution to an equation that needs factoring. For example, if the trinomial equation factors into $(x + 1)(x + 1) = 0$, then the solution will be $x = -1$—there's no need to say that the solution is $x = -1$ or -1.

Solution: Factor the trinomial and solve the two equations:

$$x^2 - 7x + 12 = 0$$
$$(x - 3)(x - 4) = 0$$
$$x - 3 = 0 \quad or \quad x - 4 = 0$$
$$x - 3 + 3 = 0 + 3 \quad x - 4 + 4 = 0 + 4$$
$$x = 3 \qquad x = 4$$

To check, put 3 in the original equation for x:

$$3^2 - 7 \cdot 3 + 12 = 0 \rightarrow 9 - 21 + 12 = 0 \rightarrow -12 + 12 = 0 \rightarrow 0 = 0$$

Put 4 in the original equation for x:

$$4^2 - 7 \cdot 4 + 12 = 0 \rightarrow 16 - 28 + 12 = 0 \rightarrow -12 + 12 = 0 \rightarrow 0 = 0$$

Therefore, 3 and 4 are the solutions to $x^2 - 7x + 12 = 0$.

Now try to solve these equations, checking your answers below.

8. Solve $x^2 - 5x - 14 = 0$.
9. Solve $x^2 + 9x + 20 = 0$.
10. Solve $x^2 - 36 = 0$.

Answers to Sample Problems

1. $(x + 8)(x + 3) = 0 \rightarrow x + 8 = 0$ or $x + 3 = 0 \rightarrow x = -8$ or $x = -3$
2. $(c - 5)(c + 1) = 0 \rightarrow c - 5 = 0$ or $c + 1 = 0 \rightarrow c = 5$ or $c = -1$
3. $(2a - 3)(a - 6) = 0 \rightarrow 2a - 3 = 0$ or $a - 6 = 0 \rightarrow a = 3/2$ or $a = 6$
4. Factor $x^2 + 9x + 20 \rightarrow (x + 5) \cdot (x + 4)$
5. Factor $x^2 - 8x + 12 \rightarrow (x - 2) \cdot (x - 6)$
6. Factor $x^2 - 6x - 16 \rightarrow (x - 8) \cdot (x + 2)$
7. Factor: $x^2 - 25 \rightarrow (x + 5) \cdot (x - 5)$
8. Solve $x^2 - 5x - 14 = 0$

$$(x - 7)(x + 2) = 0$$
$$x - 7 = 0 \quad or \quad x + 2 = 0$$
$$x = 7 \qquad x = -2$$

9. Solve $x^2 + 9x + 20 = 0$

$$(x + 5)(x + 4) = 0$$
$$x + 5 = 0 \quad or \quad x + 4 = 0$$
$$x = -5 \qquad x = -4$$

10. Solve $x^2 - 36 = 0$

$$(x - 6)(x + 6) = 0$$

$$x - 6 = 0 \quad or \quad x + 6 = 0$$

$$x = 6 \qquad\qquad x = -6$$

Hour's Up!

Based on what you've learned in Hour 10, do the following problems. Check your answers with the solution key in Appendix B.

Solve:

1. $(b - 1)(b - 9) = 0$
- **a.** $b = -1$ or 9
- **b.** $b = 1$ or -9
- **c.** $b = 1$ or 9
- **d.** $b = -1$ or -9

2. $(x - 12)(x + 4) = 0$
- **a.** $x = -12$ or -4
- **b.** $x = 12$ or -4
- **c.** $x = -12$ or 4
- **d.** $x = 12$ or 4

3. $(a + 5)(a + 5) = 0$
- **a.** $a = 0$ or -5
- **b.** $a = -5$ or 5
- **c.** $a = 5$
- **d.** $a = -5$

Factor:

4. $x^2 + 5x + 6$
- **a.** $(x + 6)(x + 1)$
- **b.** $(x + 5)(x + 1)$
- **c.** $(x + 2)(x + 3)$
- **d.** $(x + 3)(x + 3)$

Review

5. $x^2 + 6x + 5$

a. $(x + 1)(x + 4)$
b. $(x + 1)(x + 6)$
c. $(x + 2)(x + 3)$
d. $(x + 1)(x + 5)$

6. $x^2 - 3x - 18$

a. $(x - 6)(x + 3)$
b. $(x - 3)(x + 6)$
c. $(x - 2)(x - 9)$
d. $(x - 2)(x + 9)$

7. $x^2 - 9x + 20$

a. $(x + 4)(x + 5)$
b. $(x - 4)(x - 5)$
c. $(x - 4)(x + 5)$
d. $(x + 4)(x - 5)$

Solve:

8. $x^2 + 8x + 12 = 0$

a. $x = -2$ or -6
b. $x = -2$ or 6
c. $x = 2$ or -6
d. $x = 2$ or 6

9. $x^2 - 7x + 10 = 0$

a. $x = -2$ or 5
b. $x = 2$ or -5
c. $x = 2$ or 5
d. $x = -2$ or -5

10. $x^2 + 6x - 27 = 0$

a. $x = -9$ or -3
b. $x = 9$ or -3
c. $x = 9$ or 3
d. $x = -9$ or 3

Review

Part IV

Fundamentals of Fractions

Hour 11 Algebraic Fractions: Simplifying, Multiplying, and Dividing

Hour 12 Algebraic Fractions: Adding and Subtracting

Hour 13 Ratios and Proportions: Basic Percent Problems

Hour 14 Ratios and Proportions: Mixture and Work Problems

Hour 11

Algebraic Fractions: Simplifying, Multiplying, and Dividing

CHAPTER SUMMARY

In this hour you will learn about …

- Simplifying algebraic fractions by factoring
- Multiplying algebraic fractions containing exponents
- Dividing algebraic fractions

Factoring the trinomials in the previous hour's equations made it possible to solve the equations—what looked like really complex problems became much simpler when you found the factors. In much the same way, your algebra skills based on factoring are going to help you work with fractions this hour.

Simplifying Algebraic Fractions by Factoring

The division problems that you did in Hour 3 and Hour 9 gave you a strong foundation for this hour's work with algebraic fractions. As you have probably noticed, the more you study algebra, the more you realize that the same concepts keep coming up, with variations here and there, and that's very true with algebraic fractions. You'll use the factoring skills you developed in the previous hour a lot in the next few hours, and those skills will help make fractions much easier to simplify.

Consider the basic arithmetic fraction $\frac{4}{7}$. This fraction is in simplest terms—or simplified—because 1 is the only *common factor* of 4 and 7.

STRICTLY DEFINED

Common factors are numbers or variables that are factors of two or more terms. For example, 2 is a common factor of 8 and 10 because $2 \cdot 4 = 8$ and $2 \cdot 5 = 10$. 1 is a common factor of all numbers and variables because any number or variable multiplied by 1 does not change: $1 \cdot 8 = 8$ and $1 \cdot a = a$.

To say this in another way, the factors of 4 are $1 \cdot 2 \cdot 2$ and the factors of 7 are $1 \cdot 7$.

So $\frac{4}{7} = \frac{1 \cdot 2 \cdot 2}{1 \cdot 7}$. When the 1 in the numerator is divided by the 1 in the denominator, the result is 1, so you are left with $\frac{4}{7} = \frac{2 \cdot 2}{7}$, which is where you started. If the only common factor for a numerator and denominator is 1, that fraction is in simplest terms or simplified. (You don't need to show the 1 in the factored form because you know it's always there.)

Now consider the fraction $\frac{12}{15}$. Write it in factored form to see if there are any common factors other than 1:

$$\frac{12}{15} = \frac{2 \cdot 2 \cdot 3}{3 \cdot 5}$$

Yes, there's a factor of 3 in both the numerator and the denominator. Because 3 divided by 3 equals 1, this fraction can be simplified, and it becomes this:

$$\frac{12}{15} = \frac{2 \cdot 2 \cdot 3}{3 \cdot 5} = \frac{2 \cdot 2}{5} = \frac{4}{5}$$

Showing the steps that include the factors helps you understand the algebraic process at work in this problem. But after you do a few of these problems, you might see a shortcut—if you just divide both the numerator and the denominator by the common factor 3, you get the correct answer of $\frac{4}{5}$. That shortcut is fine if you use it carefully and understand why the shortcut really works, as shown in the factors of the previous fraction example.

Factoring—finding the numbers and variables that multiply to produce the numerator and denominator—is the key to simplifying arithmetic fractions, and it's also the key for algebraic fractions. Follow the factoring steps that show how to simplify these next examples, which are very similar to the monomial division problems from Hour 9:

a. $\frac{7ac}{14ab} = \frac{7ac}{2 \cdot 7ab} = \frac{c}{2b}$

b. $\frac{6x^2}{8x} = \frac{2 \cdot 3 \cdot x \cdot x}{2 \cdot 2 \cdot 2 \cdot x} = \frac{3x}{4}$

c. $\frac{10a^3b^2}{20a^4b^2} = \frac{10a^3b^2}{10 \cdot 2a^3 \cdot a \cdot b^2} = \frac{1}{2a}$

As with the basic arithmetic fractions, you are free to use a shortcut in each of the previous fractions, but be careful as you divide to follow the exponent rule and to work with the coefficients separately from the variables.

JUST A MINUTE

Here's the division rule for exponents: when dividing the same base, subtract the exponents.

The algebraic fraction examples have all been monomials, so let's look at some that include binomials or trinomials, such as this one:

$$\frac{5x^2 + 10}{5}$$

This problem could also have appeared in Hour 9. It has a binomial divided by a monomial. Recall that it was changed into two fractions and then simplified, as follows:

$$\frac{5x^2 + 10}{5} = \frac{5x^2}{5} + \frac{10}{5} = \frac{x^2}{1} + \frac{2}{1} = x^2 + 2$$

Another way to simplify this fraction is to factor the numerator and the denominator to see if there are any common factors. Factoring always involves finding what numbers or variables can be multiplied to create the expression, and with the addition sign in this fraction's numerator, you may think there is no way to factor it.

What about the distributive property?

$5(x^2 + 2) = 5x^2 + 10$ is an example of this property from Hours 4 and 6. Think of it in reverse order, like this:

$$5x^2 + 10 = 5(x^2 + 2)$$

JUST A MINUTE

With the distributive property, multiplying a sum by a number is the same as adding the product of the numbers: $a\,(b + c) = a \cdot b + a \cdot c$

You had the binomial $5x^2 + 10$—two terms added together. Now you have two factors—5 multiplied by $(x^2 + 2)$. By reversing the process of the distributive property, you have actually found the factors of $5x^2 + 10$.

Look at the original problem again:

$$\frac{5x^2 + 10}{5}$$

This time, write the problem as factors:

$$\frac{5x^2 + 10}{5} = \frac{5(x^2 + 2)}{5}$$

Because $\frac{5}{5} = 1$, the answer is $1(x^2 + 2)$, which is simply $x^2 + 2$.

The two ways to simplify the original fraction—separating into two fractions or factoring—are both acceptable, and you may choose to do the work with either method. In more complex algebraic fractions, factoring is the easier way. Let's see some examples:

d. Simplify: $\frac{4b + 8}{4b + 4c}$

Use the distributive property in reverse to factor the numerator and denominator, and then simplify:

Solution: $\frac{4b + 8}{4b + 4c} = \frac{4(b + 2)}{4(b + c)} = \frac{b + 2}{b + c}$

JUST A MINUTE

Recall that division by 0 is not possible, so the possible values for the variables in a denominator cannot include any numbers that would make the denominator equal zero. For example, if the denominator is $a - 2$, a cannot be 2. If the denominator is $x + 6$, x cannot be -6. This doesn't change the solution process, but it can be included as part of the answer and is important in higher-level math courses.

Be careful to know when to stop the simplifying process. You may be tempted to say that $b \div b$ equals 1 and that the answer is $\frac{2}{c}$. However, these b terms are actually being added to 2 and to c, so they are not being divided by each other. The previous fraction answer is $(b + 2)$ divided by $(b + c)$, and the only other way it could be written is to split it—very carefully—into the sum of two fractions with the same denominators, as follows:

$$\frac{b + 2}{b + c} = \frac{b}{b + c} + \frac{2}{b + c}$$

These two fractions cannot be simplified. So the answer to the original problem is best written as follows:

$$\frac{4b+8}{4b+4c}=\frac{b+2}{b+c}.$$

e. Simplify: $\dfrac{9a-18}{a-2}$

JUST A MINUTE

A polynomial divided by itself equals 1. For example, $(a+b)\div(a+b)=1$; $(x+y-3)\div(x+y-3)=1$.

Solution: $\dfrac{9a-18}{a-2}=\dfrac{9(a-2)}{a-2}=9$

f. Simplify: $\dfrac{3bc}{b^2c+bc^2}=\dfrac{3bc}{bc(b+c)}=\dfrac{3}{b+c}$

g. Simplify: $\dfrac{x^2+8x+15}{x^2+7x+12}$

Use your trinomial factoring skills from the previous hour to find the factors of this numerator and denominator, then simplify:

$$\frac{x^2+8x+15}{x^2+7x+12}=\frac{(x+3)(x+5)}{(x+3)(x+4)}=\frac{(x+5)}{(x+4)} \text{ or } \frac{x+5}{x+4}$$

Now try to simplify the following algebraic fractions, using the factoring method. Check your answers on page 133.

1. $\dfrac{8x+24}{x+3}=$
2. $\dfrac{9x^2+45x}{x+5}=$
3. $\dfrac{5x^4-15x^3}{5x^3}=$
4. $\dfrac{x^2+6x+8}{x^2+3x+2}=$

Being able to simplify to more complex algebraic fractions requires a high level of analytical skill that increases your ability to analyze situations in other parts of your life. Seldom, if ever, will someone ask you to factor a binomial. But the thought processes you must use to do this

work build your brainpower in many other ways. Working with algebraic fractions is definitely food for thought, and your abilities can now be extended into more complex operations, including multiplication.

Multiplying Algebraic Fractions Containing Exponents

Simplifying fractions by factoring prepares you for solving the following multiplication problem:

$$\frac{12}{18} \cdot \frac{24}{30}$$

You could multiply the terms in the numerator, then multiply the terms in the denominator, and then divide to get an answer. This would be a lot of work with some very large numbers. But factoring each numerator and denominator first allows you to simplify this problem quickly and to efficiently find the answer. Notice that 6 is a factor of each of the four numbers, so find the other factors that go with each 6:

$$\frac{12}{18} \cdot \frac{24}{30} = \frac{2 \cdot 6}{3 \cdot 6} \cdot \frac{4 \cdot 6}{5 \cdot 6} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$$

TIME SAVER

Finding the greatest common factor of the numerators and denominators lets you simplify the fractions more quickly, but any common factor can be used. For example, if 16 is in the numerator and 20 is in the denominator, using 4 as the common factor is more efficient than using 2, which requires an extra step; both approaches give the same answer eventually.

A shortcut in the previous example would be to divide each

term by 6, and you can do this as long as you watch each term carefully to see what factor remains.

A very similar process is used when you multiply more complex algebraic fractions.

Consider the problem $\frac{18x^2}{2y^3} \cdot \frac{20y}{3x^3}$. This multiplication problem is most easily simplified by factoring, dividing the common factors, and subtracting the exponents. Follow the factoring shown here:

$$\frac{18x^2}{2y^3} \cdot \frac{20y}{3x^3} = \frac{3 \cdot 6x^2}{2y^3} \cdot \frac{2 \cdot 10y}{3x^3} = \frac{6 \cdot 10}{xy^2} = \frac{60}{xy^2}$$

When you choose the factors of the larger numbers, look at the smaller numbers in the problem and try to use them as factors of the larger numbers. In the preceding example, 2 and 3 are in the denominators and they are both factors of 18, while 2 is a factor of 20. So 3 · 6 was used for the 18, and 2 · 10 for the 20.

If you had factored the 18 and 20 another way, such as 3 · 6 and 4 · 5, the end result would still be the same. Your goal is simply to find common factors, and this process may take more or fewer steps than in the example shown.

All the factors in the numerator are being multiplied, and all the factors in the denominator are also being multiplied, so the order of the factors does not change the final answer because of the commutative and associative properties.

JUST A MINUTE

Recall from Hour 4 that the commutative property allows you to multiply two algebraic factors in any order, and the answer will be the same: $a \cdot b = b \cdot a$. The associative property allows you to multiply three or more algebraic factors by grouping them in any order, and the answer will be the same: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

The next multiplication example shows factoring with the help of the distributive property in reverse order:

h. $$\frac{3x^2 + 9x}{2x} \cdot \frac{8x - 10}{3} = \frac{3x(x + 3)}{2x} \cdot \frac{2(4x - 5)}{3} = (x + 3)(4x - 5)$$

The answer can be left in binomial form, which is preferred, or you can FOIL it to create a polynomial.

Now consider a multiplication problem that includes a binomial with exponents. This unique binomial needs to be factored by the reverse of the FOIL method:

i. $$\frac{x + 5}{x^3} \cdot \frac{6x^2}{x^2 - 25} = \frac{x + 5}{x^3} \cdot \frac{6x^2}{(x - 5)(x + 5)} = \frac{6}{x(x - 5)}$$

As in example h, the answer is usually left in factored form, but you could also write it like this: $\frac{6}{x^2 - 5x}$

Now try multiplying the following fractions, checking your answers on pages 133–134.

5. $\frac{12x}{5y^3} \cdot \frac{25y^3}{2x^2} =$

6. $\frac{4x^2 + 16x}{3x} \cdot \frac{3x - 15}{2} =$

7. $\frac{x^2 - 9}{x^2} \cdot \frac{8x^3}{x - 3} =$

Multiplying and dividing are very closely linked. Your multiplying skills with simple and complex fractions will help you understand the division problems in the final part of this hour.

Dividing Algebraic Fractions

In Hour 9 you learned how to divide polynomials by monomials, the perfect introduction to your learning how to divide algebraic fractions. The main difference this hour is that the problems are more complex than they were in Hour 9, but with the help of factoring and simplifying, you'll find that dividing fractions is not difficult.

JUST A MINUTE

To divide by a number or variable, multiply by its reciprocal. To find the reciprocal, switch the expressions in the numerator and denominator: the reciprocal of 8 is $\frac{1}{8}$; the reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$.

Consider the following division problem:

j. $20x \div 4$

Recall from Hours 4 and 9 that dividing by 4 is the same as multiplying by the reciprocal of 4, which is $\frac{1}{4}$. So this problem could be written in expanded form as follows:

$$20x \div 4 = 20x \cdot \frac{1}{4} = 5 \cdot 4x \cdot \frac{1}{4} = 5x$$

A good shortcut you might use is to simply divide the coefficient 20 by 4 and get $5x$. But the expanded form shown previously gives you the process that works for simple and more complex division problems.

The next example shows several steps of factoring and simplifying. You may be able to find the answer without each small step, but don't forget that the coefficients are simplified one way, and the exponents have their own rules. Use as many of the expanded steps as you need to get the answer:

k. $\dfrac{16y}{3x^3} \div \dfrac{2y^4}{9x^2} = \dfrac{16y}{3x^3} \cdot \dfrac{9x^2}{2y^4} = \dfrac{2 \cdot 8y}{3x^3} \cdot \dfrac{3 \cdot 3x^2}{2y^4} = \dfrac{8 \cdot 3}{xy^3} = \dfrac{24}{xy^3}$

When the division problem has fractions that include polynomials, it is a good idea to first change the division to multiplication and then do the factoring:

l. $\dfrac{a^2 - 8a + 12}{a^2 - 36} \div \dfrac{3a - 6}{a + 6} = \dfrac{a^2 - 8a + 12}{a^2 - 36} \cdot \dfrac{a + 6}{3a - 6} =$

$\dfrac{(a - 2)(a - 6)}{(a + 6)(a - 6)} \cdot \dfrac{a + 6}{3(a - 2)} = \dfrac{1}{3}$

You may have been surprised that such a complex problem, with all those polynomials, would have such a simple answer. That's the power of simplifying by factoring!

Now try these division problems, checking your answers on page 134.

8. $\dfrac{25y^3}{2x} \div \dfrac{5y}{14x^3} =$

9. $\dfrac{c^2 - 16}{c^2 - 7c + 10} \div \dfrac{6c + 24}{c - 5} =$

Answers to Sample Problems

1. $\dfrac{8x + 24}{x + 3} = \dfrac{8(x + 3)}{x + 3} = 8$

2. $\dfrac{9x^2 + 45x}{x + 5} = \dfrac{9x(x + 5)}{x + 5} = 9x$

3. $\dfrac{5x^4 - 15x^3}{5x^3} = \dfrac{5x^3(x - 3)}{5x^3} = x - 3$

4. $\dfrac{x^2 + 6x + 8}{x^2 + 3x + 2} = \dfrac{(x + 2)(x + 4)}{(x + 1)(x + 2)} = \dfrac{x + 4}{x + 1}$

5. $\dfrac{12x}{5y^3} \cdot \dfrac{25y^3}{2x^2} = \dfrac{2 \cdot 6x}{5y^3} \cdot \dfrac{5 \cdot 5y^3}{2x^2} = \dfrac{30}{x}$

6. $\frac{4x^2 + 16x}{3x} \cdot \frac{3x - 15}{2} = \frac{4x(x + 4)}{3x} \cdot \frac{3(x - 5)}{2} =$

$\frac{4(x + 4)(x - 5)}{2} = 2(x + 4)(x - 5)$

7. $\frac{x^2 - 9}{x^2} \cdot \frac{8x^3}{x - 3} = \frac{(x - 3)(x + 3)}{x^2} \cdot \frac{8x^3}{x - 3} = 8x(x + 3)$

8. $\frac{25y^3}{2x} \div \frac{5y}{14x^3} = \frac{25y^3}{2x} \cdot \frac{14x^3}{5y} = \frac{5 \cdot 5y^3}{2x} \cdot \frac{2 \cdot 7x^3}{5y} = 35x^2y^2$

9. $\frac{c^2 - 16}{c^2 - 7c + 10} \div \frac{6c + 24}{c - 5} = \frac{c^2 - 16}{c^2 - 7c + 10} \cdot \frac{c - 5}{6c + 242} =$

$\frac{(c - 4)(c + 4)}{(c - 2)(c - 5)} \cdot \frac{(c - 5)}{6(c + 4)} = \frac{c - 4}{6(c - 2)}$

Review

Hour's Up!

Based on what you've learned in Hour 11, do the following problems. Check your answers with the solution key in Appendix B.

Simplify:

1. $\frac{6x + 24}{x + 4} =$

a. $x + 4$
b. 4
c. $6x$
d. 6

2. $\frac{7x^2 - 49x}{x - 7} =$

a. $7x$
b. 7
c. $x - 7$
d. x

3. $\frac{3x^5 - 18x^4}{3x^4} =$

a. 3
b. $x - 6$
c. $6x$
d. $3x$

Review

4. $\dfrac{x^2 + 7x + 12}{x^2 + 5x + 6} =$

a. $\dfrac{x + 4}{x + 2}$

b. $\dfrac{x + 2}{x + 4}$

c. $\dfrac{x + 3}{x + 2}$

d. $\dfrac{x + 4}{x + 3}$

Multiply:

5. $\dfrac{20x^4}{4y^2} \cdot \dfrac{12y}{5x^6} =$

a. $\dfrac{4}{x^2y}$

b. $\dfrac{12}{xy^2}$

c. $\dfrac{12}{xy}$

d. $\dfrac{12}{x^2y}$

6. $\dfrac{2x^2 + 8x}{5x} \cdot \dfrac{5x - 15}{4} =$

a. $\dfrac{()x + 3\ ()x - 4}{2}$

b. $\dfrac{()x\ + 4\ ()x - 3}{2}$

c. $\dfrac{()x - 4\ ()x - 3}{2}$

d. $\dfrac{()x + 4\ ()x + 3}{2}$

7. $\frac{x^2 - 16}{x^3} \cdot \frac{4x^2}{x + 4} =$

a. $\frac{x(x - 4)}{4}$

b. $\frac{4(x + 4)}{x}$

c. $\frac{4(x - 4)}{x}$

d. $\frac{x(x + 4)}{4}$

Divide:

8. $\frac{15x^2}{4y} \div \frac{5x}{24y^5} =$

a. $12x^2y^4$

b. $8xy^4$

c. $18xy$

d. $18xy^4$

9. $\frac{a^2 - 9}{a^2 - 9a + 18} \div \frac{2a + 6}{a - 3} =$

a. $\frac{a + 3}{2(a - 6)}$

b. $\frac{a - 3}{a - 6}$

c. $\frac{a - 3}{2(a - 6)}$

d. $\frac{a - 3}{2(a + 6)}$

10. $\frac{x^2 - 10x + 25}{x^2 - 16} \div \frac{2x - 10}{x + 4} =$

a. $\frac{x - 5}{2(x - 4)}$

b. $\frac{x - 4}{2(x - 5)}$

c. $\frac{x - 5}{2(x + 4)}$

d. $\frac{x - 4}{2(x + 5)}$

Review

Hour 12

Algebraic Fractions: Adding and Subtracting

CHAPTER SUMMARY

In this hour you will learn about …

- Adding algebraic fractions
- Subtracting algebraic fractions
- Finding and using least common denominators

Multiplying and dividing fractions in basic arithmetic and in algebra involves a lot of factoring. Factoring will also be helpful as you combine fractions with the last two operations—addition and subtraction.

Adding Algebraic Fractions

Let's begin with an example from arithmetic. You will see that the process used to add simple fractions applies directly to more complex algebraic fractions as well.

Consider the following addition problem:

$$\frac{7}{10} + \frac{2}{10}$$

Because the denominators are equal, this addition problem could be written as a single fraction, with the numerators summed over the denominator:

$$\frac{7}{10} + \frac{2}{10} = \frac{7+2}{10}$$

Because the 7 and the 2 are like terms, they can be added:

$$\frac{7}{10} + \frac{2}{10} = \frac{7+2}{10} = \frac{9}{10}$$

This example from basic arithmetic can be extended slightly to become an algebra problem that includes variables:

$$\frac{7x}{10} + \frac{2x}{10}$$

Again, it can be written as a single fraction and the like terms combined, as shown:

$$\frac{7x}{10} + \frac{2x}{10} = \frac{7x + 2x}{10} = \frac{9x}{10}$$

A *common denominator* is needed when two or more fractions are added.

STRICTLY DEFINED

A **common denominator** is a number, variable, or polynomial that is the same denominator for all the fractions in an addition problem.

As long as the fractions being added have equal denominators, you can combine the numerators and create a single fraction, as in the following examples:

a. $\frac{4x}{15} + \frac{7x}{15} = \frac{4x + 7x}{15} = \frac{11x}{15}$

$11x$ and 15 have no common factors, so $\frac{11x}{15}$ is the simplified answer.

PROCEED WITH CAUTION

Always check the answer after you form the single fraction. To be simplified, any factors common to the numerator and denominator must be divided out. You can do this division mentally or show the steps of factoring the terms, as you learned in Hour 11.

b. $\frac{4x}{15} + \frac{x}{15} = \frac{5x}{15} = \frac{x}{3}$

$5x$ and 15 have a common factor of 5, so each term must be divided by 5 to give the simplified answer of $\frac{x}{3}$.

c. $\frac{3a^2}{10} + \frac{7a^2}{10} + \frac{a^2}{10} = \frac{11a^2}{10}$

$11a^2$ and 10 have no common factors, so $\frac{11a^2}{10}$ is the simplified answer.

In algebra, unlike basic arithmetic, it's acceptable to leave a fraction with

a numerator larger than the denominator. Common factors do need to be divided out, though, as you see in the next example:

d. $\frac{5x^3}{12y} + \frac{7x^3}{12y} + \frac{3x^3}{12y} = \frac{15x^3}{12y} = \frac{5x^3}{4y}$

$5x^3$ and $4y$ have no common factors, so $\frac{5x^3}{4y}$ is the simplified answer.

Now try these addition problems, checking your answers on pages 143–144.

1. $\frac{8x}{3} + \frac{7x}{3} =$
2. $\frac{5x^2}{12} + \frac{3x^2}{12} =$
3. $\frac{2a}{5b} + \frac{4a^2}{5b} + \frac{a}{5b} =$

Recall from Hours 4 and 7 that subtracting is closely related to adding. As you learn how to subtract fractions, you'll see that the process is almost identical to the addition problems in the earlier section. However, be prepared to watch for changing signs, just as you did when you subtracted polynomials.

Subtracting Algebraic Fractions

Consider the following subtraction problem:

$$\frac{3}{7a} - \frac{5}{7a}$$

JUST A MINUTE

Subtraction can be changed to addition if you change the signs of all the terms in the second expression: $(12x + 7) - (5x - 8) = 12x + 7 + (-5x) + (+8) = 7x + 15$

Because the denominators are equal, the two numerators can be combined to form a single fraction:

$$\frac{3}{7a} - \frac{5}{7a} = \frac{3}{7a} + \frac{-5}{7a} = \frac{3 + (-5)}{7a} = \frac{-2}{7a}$$

-2 and $7a$ have no common factors, so $\frac{-2}{7a}$ is the simplified answer.

Watch the signs carefully if the first term is negative:

$$\frac{-6x}{5} - \frac{9x}{5} = \frac{-6x}{5} + \frac{-9x}{5} = \frac{-6x + (-9x)}{5} = \frac{-15x}{5} = \frac{-3x}{1} = -3x$$

–15 and 5 have a common factor of 5. Dividing both terms by 5 gives the simplified answer of $-3x$.

In the following example, the second numerator is a binomial, but the subtraction process is the same:

$$\frac{3b}{10} - \frac{6b + 5}{10}$$

By changing the signs in the binomial $6b + 5$, this subtraction problem becomes an addition problem, as shown here:

$$\frac{3b}{10} - \frac{6b + 5}{10} = \frac{3b}{10} + \frac{-6b - 5}{10}$$

Adding the fractions gives the following solution:

$$\frac{3b}{10} - \frac{6b + 5}{10} = \frac{3b}{10} + \frac{-6b - 5}{10} = \frac{3b + -6b - 5}{10} = \frac{-3b - 5}{10}$$

$-3b$, –5, and 10 have no common factors, so $\frac{-3b - 5}{10}$ is the simplified answer.

PROCEED WITH CAUTION

When you add or subtract fractions, the numerators can include polynomials. Only the like terms can be added or subtracted to give a single term: $9x$ and $5x$ are like terms and combine to $14x$; $9x$ and 5 are not like terms and cannot be combined into a single term—their sum is written $9x + 5$.

When both numerators are binomials, changing the signs in the second one allows you to add the fractions:

$$\frac{2x - 7}{14x^2} - \frac{3x - 7}{14x^2} = \frac{2x - 7}{14x^2} + \frac{-3x + 7}{14x^2} = \frac{2x - 7 + (-3x) + 7}{14x^2} = \frac{-x}{14x^2} = \frac{-1}{14x}$$

$-x$ and $14x^2$ have a common factor of x. Dividing both terms by x gives the simplified answer of $\frac{-1}{14x}$.

Now try the following problems, checking your answers on page 144.

4. $\frac{3}{8x} - \frac{7}{8x} =$

5. $\dfrac{-9a^3}{16} - \dfrac{5a^3}{16} =$

6. $\dfrac{5x}{13y^2} - \dfrac{-4}{13y^2} =$

In all the addition and subtraction problems so far, the fractions had common denominators. Combining the numerators allowed you to create a single fraction. But what if the denominators are not equal? Your factoring skills will help you get each problem into a new form, and then you can do the adding and subtracting.

Finding and Using Least Common Denominators

When the denominators in fractions are not equal, those fractions cannot be added or subtracted. Consider the following addition problem:

$$\frac{7}{2} + \frac{9}{10}$$

These two fractions cannot be combined into one single fraction—at least, not the way they are now. If you can change one denominator to equal the other, then you can combine them into one fraction.

Think about the two denominators, 2 and 10. If you multiply 2 times 5, you get 10. However, multiplying only the denominator 2 by 5 would change the value of the first fraction. What if you multiply both the numerator and denominator by 5? It would look like this:

$$\frac{7 \cdot 5}{2 \cdot 5}$$

Because $\dfrac{5}{5} = 1$, you are really just multiplying the original fraction by 1, which doesn't change its value. The addition problem would become this:

$$\frac{7 \cdot 5}{2 \cdot 5} + \frac{9}{10} = \frac{35}{10} + \frac{9}{10}$$

This new addition problem is easily solved:

$$\frac{35}{10} + \frac{9}{10} = \frac{44}{10} = \frac{22}{5}$$

You began with the denominators of 2 and 10. Knowing that 2 is a factor of 10 allowed you to see that multiplying by the missing factor of

5 would give you the common denominators you need when adding or subtracting. The factoring process makes it all possible.

TIME SAVER

To find the common denominator, list the factors of the given denominators. Using the factors that are in each denominator will lead you to the least common denominator (LCD).

The following addition problem also has different denominators:

$$\frac{9}{4} + \frac{5}{6}$$

Because 4 is not a factor of 6, you need to find a new number that both 4 and 6 are factors of. There are many ways to do this, but the easiest is to multiply the larger number, 6, by 2, which gives 12. Is 4 a factor of 12? Yes. 4 times 3 is 12.

Multiplying the numerator and denominator of $\frac{5}{6}$ by 2 and the numerator and denominator of $\frac{9}{4}$ by 3, you have found the least common denominator, which is 12. You also have two fractions that can easily be combined into one:

$$\frac{9 \cdot 3}{4 \cdot 3} + \frac{5 \cdot 2}{6 \cdot 2} = \frac{27}{12} + \frac{10}{12} = \frac{27 + 10}{12} = \frac{37}{12}$$

37 and 12 have no common factors, so $\frac{37}{12}$ is the simplified answer.

The factoring process from the previous examples works in all problems that need common denominators, including those with variables, as in the next example:

$$\frac{8}{5x^2} + \frac{7}{20x^2}$$

The denominators already have common factors of x^2 that are equal, but the coefficients, 5 and 20, are not. 5 times 4 is 20, so 4 is multiplied by the original numerator and denominator in the first fraction:

$$\frac{8 \cdot 4}{5x^2 \cdot 4} + \frac{7}{20x^2} = \frac{32}{20x^2} + \frac{7}{20x^2} = \frac{39}{20x^2}$$

39 and $20x^2$ have no common factors, so $\frac{39}{20x^2}$ is the simplified answer.

TIME SAVER

You do not need to show all the detailed solution steps. They are included here as you learn the algebraic process. Combine some of the steps, if you prefer.

The following problem shows different coefficients and variables in the denominators of a subtraction problem, but finding the common factors helps you find the LCD:

$$\frac{2a}{3b^3} - \frac{5}{7b}$$

3 and 7 have no common factors. Multiplying the larger number, 7, by 2 doesn't help, since 3 is not a factor of 14. Multiplying 7 by 3 works, so 21 is part of the LCD. For the variables, b is a factor of b^3 because b times b^2 equals b^3. The LCD for this problem is $21b^3$.

The first fraction's numerator and denominator need to be multiplied by 7, and the second fraction's numerator and denominator need to be multiplied by $3b^2$. The work is shown here:

$$\frac{2a \cdot 7}{3b^3 \cdot 7} - \frac{5 \cdot 3b^2}{7b \cdot 3b^2} = \frac{14a}{21b^3} - \frac{15b^2}{21b^3} = \frac{14a + (-15b^2)}{21b^3}$$

None of the terms in the fraction have common factors, so $\frac{14a + (-15b^2)}{21b^3}$ is the simplified answer.

The previous problem is more complex, but the process works the same as in the easier ones. Understanding and using the algebraic processes is still your goal, no matter how complex the problem is.

Now try the following problems, checking your answers on page 144.

7. $\frac{9}{10x^2} + \frac{3}{20x^2} =$
8. $\frac{5}{9a^3} - \frac{3}{2a} =$

Answers to Sample Problems

1. $\frac{8x}{3} + \frac{7x}{3} = \frac{8x + 7x}{3} = \frac{15x}{3} = \frac{5x}{1} = 5x$
2. $\frac{5x^2}{12} + \frac{3x^2}{12} = \frac{5x^2 + 3x^2}{12} = \frac{8x^2}{12} = \frac{2x^2}{3}$

3. $\frac{2a}{5b} + \frac{4a^2}{5b} + \frac{a}{5b} = \frac{2a + 4a^2 + a}{5b} = \frac{4a^2 + 3a}{5b}$

4. $\frac{3}{8x} - \frac{7}{8x} = \frac{3 + (-7)}{8x} = \frac{-4}{8x} = \frac{-1}{2x}$

5. $\frac{-9a^3}{16} - \frac{5a^3}{16} = \frac{-9a^3 + (-5a^3)}{16} = \frac{-14a^3}{16} = \frac{-7a^3}{8}$

6. $\frac{5x}{13y^2} - \frac{-4}{13y^2} = \frac{5x + (+4)}{13y^2} = \frac{5x + 4}{13y^2}$

7. $\frac{9}{10x^2} + \frac{3}{20x^2} = \frac{9 \cdot 2}{10x^2 \cdot 2} + \frac{3}{20x^2} = \frac{18}{20x^2} + \frac{3}{20x^2} = \frac{21}{220^2}$

8. $\frac{5}{9a^3} - \frac{3}{2a} = \frac{5 \cdot 2}{9a^3 \cdot 2} - \frac{3 \cdot 9a^2}{2a \cdot 9a^2} = \frac{10}{18a^3} - \frac{27a^2}{18a^3} = \frac{10 + (-27a^2)}{18a^3}$

Review

Hour's Up!

Based on what you've learned in Hour 12, do the following problems. Check your answers with the solution key in Appendix B.

Simplify:

1. $\frac{6x^3}{5y^2} + \frac{9x^3}{5y^2} =$

- **a.** $\frac{9x^3}{y^2}$
- **b.** $\frac{3x^3}{y^2}$
- **c.** $\frac{3x^6}{y^4}$
- **d.** $\frac{15x^3}{10y^2}$

2. $\frac{8a^2}{15b} + \frac{3a^2}{15b} + \frac{a^2}{15b} =$

- **a.** $\frac{11a^2}{15b}$
- **b.** $\frac{4a^2}{15b}$
- **c.** $\frac{4a^6}{5b^3}$
- **d.** $\frac{4a^2}{5b}$

Review

3. $\frac{5x^4}{7y^3} + \frac{6x^4}{7y^3} + \frac{3x^4}{7y^3} =$

a. $\frac{2x^4}{y^3}$

b. $\frac{14x^4}{21y^3}$

c. $\frac{2x^{12}}{y^9}$

d. $\frac{7x^4}{y^3}$

4. $\frac{12c}{5b} - \frac{17c}{5b} =$

a. $\frac{-5c}{b}$

b. $\frac{5c}{b}$

c. $\frac{c}{b}$

d. $\frac{-c}{b}$

5. $\frac{8y^2}{9x^3z} - \frac{14y^2}{9x^3z} =$

a. $\frac{-6y^2}{3x^3z}$

b. $\frac{-2y^2}{3x^3z}$

c. $\frac{-y^2}{18x^3z}$

d. $\frac{-2y^4}{3x^6z^2}$

6. $\frac{-9}{4c^2} - \frac{-11}{4c^2} =$

a. $\frac{20}{4c^2}$

b. $\frac{2}{c^2}$

c. $\frac{1}{2c^2}$

d. $\frac{-2}{c^2}$

7. $\frac{7}{12x^2} + \frac{5}{3x^2} =$

a. $\frac{12}{4x^2}$

b. $\frac{27}{4x^2}$

c. $\frac{9}{4x^2}$

d. $\frac{9}{12x^2}$

8. $\frac{5}{4d^2} - \frac{7}{4d} =$

a. $\frac{-2d}{4d^2}$

b. $\frac{-d}{2d^2}$

c. $\frac{-1}{2d}$

d. $\frac{5 - 7d}{4d^2}$

9. $\frac{1}{6a} + \frac{5}{2a^3} =$

a. $\frac{a^2 + 15}{6a^3}$

b. $\frac{a^2 + 5}{6a^3}$

c. $\frac{a^2 + 5}{2a^3}$

d. $\frac{6}{12a^3}$

10. $\frac{10x}{3y^2} - \frac{4}{3y} =$

a. $\frac{10x - 4}{3y^2}$

b. $\frac{10x - 4y}{3y^2}$

c. $\frac{6xy}{3y^2}$

d. $\frac{2}{y}$

Review

Hour 13

Ratios and Proportions: Basic Percent Problems

CHAPTER SUMMARY

In this hour you will learn about …

- Setting up and simplifying ratios to solve problems
- Using proportions
- Setting up and solving percent problems

You have been doing a lot of heavy lifting the past few hours, working with complex polynomials and lots of operations and fractions. Most of it has been leaning toward the theoretical side of the spectrum, with the focus on polynomial expressions and few real-life problems.

This hour you return to the real world and discover the many ways algebra can be helpful in comparing everything from student/teacher ratios at different schools to the percentage of sugar in different brands of breakfast cereal.

Setting Up and Simplifying Ratios to Solve Problems

Comparing two numbers often means knowing which is greater than the other, such as when you want to know if your favorite team won the last game. Did your team score more points than the other team? On another level, you learned all about plotting numbers on the real number line in Hour 2, and the > and < symbols give you an efficient way to describe the relationship between two numbers.

Many times, though, you need to know how two numbers compare as a *ratio*, which brings you back to the world of fractions—not theoretical ones with polynomial terms, but more practical ones with much simpler

numbers and variables. A good example from sports is a baseball player's batting average: the number of times the player gets on base after hitting the ball is the numerator, and the number of times the player has been at bat is the denominator. This fraction is then changed into a decimal. This comparison process is what ratios are all about.

STRICTLY DEFINED

A **ratio** is a fraction that comes from dividing one number by another as a way to compare the two numbers.

Ratios can be written in different forms, but they are all basically fractions. The first number or variable named is the numerator, and the second is the denominator.

The ratio of men to women in a meeting could be written one of the following three ways:

- Men to W men
- Men:Women
- $\frac{\text{Men}}{\text{Women}}$

If there are 15 men and 19 women, these expressions would become:

- 15 to 19
- 15:19
- $\frac{15}{19}$

GO TO ◄

Recall that writing an algebraic or a basic fraction in simplest terms means finding the common factors of the numerator and denominator and dividing both the numerator and denominator by those common factors. Refer to Hour 11 for examples of this process.

Ratios often compare different types of quantities, such as men to women or miles traveled to gallons of gas used—you are not limited to comparing things that are the same. If you are finding the ratio of the cost of a loaf of white bread to a loaf of wheat bread, the terms are both in dollars. Your ability to simplify algebraic fractions by factoring helps you write the ratios in simplest terms. Whether the ratio includes numbers, variables, or both, be sure to simplify completely, as shown in the following examples:

a. Find the student/teacher ratio for the English department at the local community college that has 600 students taking English classes and 20 English teachers.

Solution: The fraction form is generally preferred. The first item listed (students) becomes the numerator, and the second (teachers) becomes the denominator.

$\frac{600}{20}$ is the ratio, but this fraction has common factors in the numerator and denominator and needs to be simplified.

10 is one common factor. Dividing 600 by 10 gives 60, and dividing 20 by 10 gives 2. The new fraction is this:

$$\frac{60}{2}$$

2 is a common factor, and dividing both terms by 2 gives this:

$$\frac{30}{1}$$

So the student/teacher ratio is 30 to 1.

This information can be used in many ways, but it's important to realize that ratios come from numbers that are put in fraction form and then simplified. However, knowing that the ratio is 30 to 1 tells you nothing about the number of students in any particular English class. In fact, there may be 30 students in each of the 20 teachers' classes, or it's possible there are 90 students in some and 10 in others. Be aware of this as you work with ratios.

b. Write the following ratio in simplest form:

$36x{:}26x$

Solution: In fraction form, this ratio becomes:

$$\frac{36x}{26x}$$

Find the common factors (2 and x) and divide the numerator and denominator by the common factors to put this ratio in simplest terms.

$$\frac{36x}{26x} = \frac{18}{13}$$

The simplified ratio is 18 to 13, 18:13, or $\frac{18}{13}$.

c. Write the ratio $\frac{10xy^3}{35x^4y^2}$ in simplest terms:

You may recall this type of problem—as well as the one in example b—from your work with dividing polynomials and simplifying algebraic fractions. Good! Your skills are coming together to help you see the overlap among the different topics in algebra. You will use those prior skills to do many of the problems in this section.

Solution: Find and divide by the common factors, which are 5, x, and y^2.

$$\frac{10xy^3}{35x^4y^2} = \frac{2y}{7x^3}$$

JUST A MINUTE

When simplifying algebraic fractions with exponents on the variables, write the final answer with positive exponents. The *x* variable in example c could have a –3 exponent, but keeping the *x* variable in the denominator makes the exponent +3.

When making ratios to compare quantities, in some cases you must convert the unit measure of one to the other. For example, if you spend a total of 120 minutes talking on the phone on a given day and a total of 3 hours in meetings on the same day, you can find the ratio of phone time to meeting time. However, you cannot use 120 to 3 because minutes and hours are different ways to measure the same thing: time. You need to change the hours to minutes or the minutes to hours—the choice is yours. Both solution methods are shown in the next example.

d. Write the ratio of phone time to meeting time for a day when you spend 120 minutes on the phone and 3 hours in meetings:

120 minutes:3 hours

Solution A: 120 minutes equals 2 hours, so the ratio of hours to hours can be written in fraction form as follows:

$$\frac{2}{3}$$

Solution B: 3 hours equals 180 minutes, so the ratio of minutes to minutes can be written in fraction form as follows:

$$\frac{120}{180}$$

TIME SAVER

Simplifying fractions—dividing the numerator and denominator by the common factor—can be done in small steps with small common factors or more quickly with the greatest common factor. Either way, the final answer will be the same. A calculator can help with the division process, especially when the fraction contains larger numbers.

This fraction's tems have a common factor of 10, so divide both by 10.

$$\frac{120}{180} = \frac{12}{18}$$

These terms have a common factor of 6. Dividing both by 6 gives this:

$$\frac{120}{180} = \frac{12}{18} = \frac{2}{3}$$

With either method, the solution is $\frac{2}{3}$, so the ratio of time you spent on the phone to meetings that day was 2 to 3.

If you had used 120:3, the ratio would be $\frac{120}{3}$. Dividing by the common factor of 3 gives this:

$$\frac{120}{3} = \frac{40}{1}$$

Certainly, that is a very different answer! And it's the wrong answer.

Making ratios of different things—women to men, miles to gallons, students to teachers—gives accurate answers. However, ratios of the same type of measures—time, weight, and distance, to name a few—have to be in the same type of units to give accurate answers.

Now try to write the following ratios in simplest terms, checking your answers on page 160.

1. $8a^2:24a^2$ =
2. 5 hours:60 minutes =
3. 18 days:2 weeks =

When you know how things compare as a ratio, you can set up equations that help you solve specific problems. Let's go back to the example of the 600 students who are taking English at the community college.

e. If the ratio of men to women taking English is 3 to 7, what is the number of women taking English?

Solution: To write the equation for this problem, you have to work a bit backward through the ratio process.

The ratio of men to women is 3 to 7, which is $\frac{3}{7}$. This ratio $\frac{3}{7}$ could have come from $\frac{3a}{7a}$. This is helpful because $\frac{3a}{7a}$ could be written in words as $3a$ to $7a$. The number of men ($3a$) and women ($7a$) in English (600 students) is the sum of $3a$ and $7a$, so the equation could be written as follows:

GO TO ◄
Refer to Hour 5 for additional examples of solving equations with addition, subtraction, multiplication, and division.

$$3a + 7a = 600$$
$$10a = 600$$
$$a = 60$$

Because $3a$ is the number of men, $3 \cdot 60$ equals 180 men. The number of women is $7a$, and $7 \cdot 60$ equals 420 women. In this example, the variable is a, but you may use any variable you prefer, as long as the same one appears throughout the equation.

Now try these word problems, checking your answers on page 160.

4. The ratio of women to men attending the conference was 3 to 4. How many of the 1,400 attendees were men?
5. The football team's ratio of wins to losses was 5 to 4 last year. How many of their 27 games did they win?

Ratios are a very efficient way to compare two quantities. When two ratios are equal, the real-life applications of these equations are almost limitless, as you will see in the next part of Hour 13.

Using Proportions

If you need 4 cups of flour to make 6 dozen of your favorite cookies, how much flour will you need for 9 dozen cookies? A *proportion* gives you the equation you solve to find the answer.

STRICTLY DEFINED

A **proportion** is an equation made of two ratios that are equal to each other.

The flour in a cookie recipe must be in the correct ratio with the other ingredients to make the cookies taste right. If you want to make more or fewer cookies than 6 dozen, the amount of flour—and the other ingredients—must be adjusted. This real-life situation is a good example of how proportions are used everywhere from home to work, sports to finance, medical research to shopping.

The ratio in the flour problem comes from 4 cups of flour to 6 dozen cookies. This ratio in fraction form is written as follows:

$$\frac{4}{6}$$

Dividing the numerator and denominator by the common factor 2 simplifies the fraction to this:

$$\frac{4}{6} = \frac{2}{3}$$

The ratio of cups of flour to dozens of cookies is 2:3. This can also be described as "2 cups of flour for every 3 dozen cookies."

It takes 2 cups of flour to make 3 dozen cookies, and the ratio of the flour to the number of cookies (in dozens) must be the same, no matter how may cookies you want to bake. So the proportion for finding how much flour (f) is needed to make 9 dozen cookies is this:

$$\frac{2}{3} = \frac{f}{9}$$

PROCEED WITH CAUTION

The ratios that form proportions must be written in the same order; if the first ratio is cups of flour:dozens of cookies, the second one must also be cups of flour:dozens of cookies.

This proportion equation is not very complex, but it also doesn't look like the equations you learned about in Hour 5.

The best way to solve a proportion is to multiply in an "x" pattern across the equals sign, which leads to a new equation. The variable can then be isolated. The method is shown here:

$$\frac{2}{3} = \frac{f}{9} \rightarrow 2 \cdot 9 = 3 \cdot f \rightarrow 18 = 3 \cdot f$$

Divide both sides of this equation by 3:

$$18 \div 3 = 3f \div 3 \rightarrow 6 = f$$

You will need 6 cups of flour to make 9 dozen cookies.

All proportions, including those with polynomial terms, can be solved by this method, often called the "cross-product." This is the most efficient way to solve any equation made of one fraction that equals another fraction, as in the following example:

f. Solve for x: $\frac{3x}{8} = \frac{9}{4}$

Solution: The cross-product results in a new equation:

$$\frac{3x}{8} = \frac{9}{4} \rightarrow 3x \cdot 4 = 9 \cdot 8 \rightarrow 12x = 72$$

Divide both sides by 12:

$$12x \div 12 = 72 \div 12$$
$$x = 6$$

To check, put 6 in for x in the original proportion:

$$\frac{3x}{8} = \frac{9}{4} \rightarrow \frac{3 \cdot 6}{8} = \frac{9}{4} \rightarrow 18 \cdot 4 = 9 \cdot 8 \rightarrow 72 = 72;$$

therefore, $x = 6$ is the solution.

g. Solve for a: $\frac{4}{a-5} = \frac{2}{3}$

JUST A MINUTE

More complex proportions can include binomials, which can be simplified by the distributive property: $x \cdot (y + z) = x \cdot y + x \cdot z$ or $x \cdot (y + z) \cdot x = x \cdot y + x \cdot z$

Solution: The cross-product results in a new equation:

$$\frac{4}{a-5} = \frac{2}{3} \rightarrow 4 \cdot 3 = 2 \cdot (a - 5) \rightarrow 12 = 2a - 10$$

To help isolate the variable, add 10 to both sides:

$$12 + 10 = 2a - 10 + 10 \rightarrow 22 = 2a$$

Dividing both sides by 2 solves the equation:

$$22 \div 2 = 2a \div 2 \rightarrow 11 = a$$

To check, put 11 in for a in the original proportion:

$$\frac{4}{a-5} = \frac{2}{3} \rightarrow \frac{4}{11-5} = \frac{2}{3} \rightarrow \frac{4}{6} = \frac{2}{3} \rightarrow \frac{4 \div 2}{6 \div 2} = \frac{2}{3} \rightarrow \frac{2}{3} = \frac{2}{3}$$

Therefore, $a = 11$ is the solution.

Now try to solve these proportions, checking your answers on page 160.

6. $\frac{2c}{3} = \frac{8}{6}$
7. $\frac{b-3}{8} = \frac{3}{4}$
8. It took you 30 minutes to write 12 thank-you notes. How long will it take to write 18 notes at the same rate?

This hour began with finding the student/teacher ratio for the English classes at a community college. You also learned how to solve proportions that include a variable. Now let's move to the hour's final goal: comparing the percent of sugar per serving in different breakfast cereals. All of these situations have ratios as their common element.

Setting Up and Solving Percent Problems

Earlier this hour you found that the ratio of men to women in a meeting was 15:19. How does that compare to another meeting, where the ratio of men to women was 5:7? Would writing the ratios in fraction form help? How does $\frac{15}{19}$ compare to $\frac{5}{7}$?

JUST A MINUTE

Recall that a fraction can always be written as a division problem: $\frac{6}{7}$ is the same as 6÷7; $\frac{14}{3}$ is the same as 14÷3.

Comparing two ratios or fractions can be confusing, especially when the denominators are not the same. You could try to find a common denominator, but there's an easier way, especially with the help of a calculator. You need a basic one with the +, –, ·, and ÷ operations for the rest of this hour's work.

The fractions $\frac{15}{19}$ and $\frac{5}{7}$ are really just division problems and could be written 15 ÷ 19 and 5 ÷ 7. If you had to divide 15 by 19 or 5 by 7, writing out the long division would take a lot of time. Maybe you have even forgotten how to do a long-division problem!

A calculator can take the numbers and operations that you put in it and do all the work for you. Let's do 15 ÷ 19 on your calculator:

Find the ÷ symbol on your calculator, but don't touch it yet. First you need to enter the 15 (the numerator of our fraction), then touch the ÷ symbol, then enter the 19, and then touch the = button.

You should see 0.78947… in the display. The calculator has changed $\frac{15}{19}$ into a decimal, and 0.78947… is .79 when rounded to the nearest hundredth.

The decimal number .79, which is 79 hundredths, can be written as a fraction: $\frac{79}{100}$

JUST A MINUTE

Remember that numbers have place values. For example, the first number to the right of the decimal point has a place value of tenths. The second number to the right of the decimal point has a place value of hundredths.

Now you know that $\frac{15}{19}$ is very close to being $\frac{79}{100}$.

What is 5 ÷ 7? Use the calculator again for this, and you should see 0.71428… in the display, which is .71, or 71 hundredths, when you round it off to the nearest hundredth.

So $\frac{5}{7}$ is very close to being $\frac{71}{100}$.

Clearly, $\frac{79}{100}$ is greater than $\frac{71}{100}$. You now know that $\frac{15}{19}$ is greater than $\frac{5}{7}$.

The beauty of working with the decimal form of these fractions is that you can see right away which is smaller or larger—they could also be equal.

The next step is to learn how *percents* fit into your knowledge of fractions and decimals.

STRICTLY DEFINED

The word **percent** means hundredths: eight percent means eight hundredths. This can also be written: 8% means .08 or 8/100.

The fractions you compared, $\frac{15}{19}$ and $\frac{5}{7}$, were changed into decimals with your calculator's help. Their decimal numbers were rounded to the nearest hundredth because percents are another way to look at hundredths, as shown in the following examples:

h. Nine hundredths = $\frac{9}{100} = .09 = 9\%$

i. 37 hundredths = $\frac{37}{100} = .37 = 37\%$

Any fraction can be changed into a percent by first being written in decimal form, using a calculator for the division, if necessary. The earlier examples of $\frac{15}{19}$ and $\frac{5}{7}$ could be written in percent form as 79% and 71%.

Because percents are used so often in many calculations related to business, medicine, and finance, you need to know not only how to write fractions as percents, but also how to use percents correctly. The key is to always think of a percent problem as a proportion.

Although you will encounter three types of percent problems, one simple proportion works for them all:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100}$$

Follow the solution steps for the next three examples to see how this proportion helps you answer the questions:

j. 16 is 20% of what number?

The "is" in this problem equals 16; the "of" in this problem equals the unknown variable, so use *x;* and the % in this problem equals 20. Putting all these terms into the proportion creates the following:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{16}{x} = \frac{20}{100}$$

Recall that proportions can be solved with the cross-product method, which gives an equation that can be solved by dividing both sides by 20:

$$1{,}600 = 20 \cdot x$$
$$80 = x$$

From this solution, you know that 16 is 20% of 80.

If you have done percent problems before, you may have learned different methods for the different types of percent problems. But this simple proportion $\frac{\text{is}}{\text{of}} = \frac{\%}{100}$ works for all three. You just need to know what

each of the terms are in your specific problem: the "is," the "of," and the %. The 100 term is always a constant and doesn't change. Another nice feature of this proportion is that you don't move the decimal point in the percent, which does happen in some other methods. 75% becomes 75 over 100 in the proportion.

k. What percent of 24 is 36? (Or, stated another way, 36 is what percent of 24?)

Solution: The "is" in this problem equals 36; the "of" in this problem equals 24, and the % in this problem equals the unknown, so use x. Putting all these terms into the proportion creates the following:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{36}{24} = \frac{x}{100}$$

Using the cross-product method gives an equation that can be solved by dividing both sides by 24:

$$3{,}600 = 24 \cdot x$$
$$150 = x$$
$$x = 150$$

From this solution, you know that 150% of 24 is 36. Yes, percent answers can be more than 100. Be sure that, if you are trying to find a percent, you remember to put the % symbol after your answer.

The third example is possibly the most common percent problem, and it works nicely with the proportion method.

TIME SAVER

Another efficient way to find the percent of a number is to change the percent to a decimal and multiply it times the number. What is 30% of 80? 30% = .30; Multiplying .30 times 80 equals 24. So 30% of 80 = 24. The percent proportion method may be easier because it does not require changing the percent to a decimal.

l. What is 80% of 40?

The "is" in this problem equals the unknown variable, so use x; the "of" in this problem equals 40, and the % in this problem equals 80. Putting all these terms into the proportion creates the following:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{x}{40} = \frac{80}{100}$$

Using the cross-product method gives an equation that can be solved by dividing both sides by 100:

$$100x = 40 \cdot 80$$
$$100x = 3{,}200$$
$$x = 32$$

From this solution, you know that 32 is 80% of 40.

No matter what part of a percent problem you need to find, you can use the proportion method to solve for the unknown.

Now try to solve the following percent problems, checking your answers on page 160.

9. 42 is 60% of what number?
10. What percent of 80 is 60?
11. What is 90% of 70?

To finish this hour, let's use percents to compare the sugar in different brands of cereal. The nutrition label on each box lists percents of daily value for many of the nutrition facts, which is helpful because that lets you compare different amounts of nutrients directly.

But the sugar is not listed as a percent—it's shown in amount per serving—which makes it difficult to compare one cereal's sugar content to another. Percents allow you to make the comparison. Here's how.

One brand of granola has 12 grams of sugar in a half-cup (55 grams) serving, and another brand has 10 grams of sugar in a third-cup (37 grams) serving.

What percent of 55 is 12? What percent of 37 is 10? Using the percent proportion, the solution steps follow:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{12}{55} = \frac{x}{100} \rightarrow 1200 = 55x \rightarrow x = 21.8\%$$

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{10}{37} = \frac{x}{100} \rightarrow 1000 = 37x \rightarrow x = 27\%$$

Changing the ratios to percents lets you compare the sugar in each cereal. You might be surprised to see that 12 grams of sugar in a half-cup serving is actually a lower percentage than 10 grams in a third-cup serving.

Next hour you will learn more percent applications, including how to find percent of change for both increase and decrease. This hour, you saw the power of proportions and their usefulness in so many specific real-life situations. The basic math computations involved in solving these problems also require algebra skills for isolating variables and finding practical solutions. The fusion of math and algebra is probably most evident in these ratio, proportion, and percent problems.

Answers to Sample Problems

1. $8a^2 : 24a^2 = \frac{8a^2}{24a^2} = \frac{1}{3}$
2. 5 hours:60 minutes. Changing the 60 minutes to 1 hour, the ratio becomes 5 hours:1 hour $\rightarrow \frac{5}{1}$.
3. 18 days:2 weeks. Changing the 2 weeks to 14 days, the ratio becomes 18 days:14 days $\rightarrow \frac{18}{14} = \frac{9}{7}$.
4. women:men $\rightarrow$ 3:4; $3a + 4a = 1{,}400 \rightarrow 7a = 1{,}400 \rightarrow a = 200$. To find the number of men, which is $4a$, put 200 in for $a \rightarrow 4 \cdot 200 = 800$; there were 800 men attending the conference.
5. wins:losses = 5:4; $5a + 4a = 27 \rightarrow 9a = 27 \rightarrow a = 3$. To find the number of wins, which is $5a$, put 3 in for $a \rightarrow 5 \cdot 3 = 15$; the team won 15 games.
6. $\frac{2c}{3} = \frac{8}{6} \rightarrow 12c = 24 \rightarrow c = 2$
7. $\frac{b-3}{8} = \frac{3}{4} \rightarrow 4(b - 3) = 3 \cdot 8 \rightarrow 4b - 12 = 24 \rightarrow 4b - 12 + 12 = 24 + 12 \rightarrow 4b = 36 \rightarrow b = 9$
8. $\frac{30}{12} = \frac{x}{18} \rightarrow 30 \cdot 18 = 12 \cdot x \rightarrow 540 = 12x \rightarrow 540 \div 12 = 12x \div 12 \rightarrow 45 = x$. It will take you 45 minutes at the same rate.
9. $\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{42}{x} = \frac{60}{100} \rightarrow 4200 = 60x \rightarrow x = 70$; 42 is 60% of 70
10. $\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{60}{80} = \frac{x}{100} \rightarrow 6000 = 80x \rightarrow x = 75\%$; 75% of 80 is 60
11. $\frac{\text{is}}{\text{of}} = \frac{\%}{100} \rightarrow \frac{x}{70} = \frac{90}{100} \rightarrow 100x = 6300 \rightarrow x = 63$; 63 is 90% of 70

Hour's Up!

Review

Based on what you've learned in Hour 13, do the following problems. Check your answers with the solution key in Appendix B.

Simplify the following ratios:

1. $32a^4:4a^2$
- **a.** 8
- **b.** $8a$
- **c.** $8a^2$
- **d.** $8a^3$

2. 50 minutes:5 hours
- **a.** $\frac{1}{6}$
- **b.** $\frac{1}{10}$
- **c.** $\frac{50}{1}$
- **d.** $\frac{10}{1}$

3. 14 oz.:3 lbs. (1 lb. = 16 oz.)
- **a.** $\frac{14}{3}$
- **b.** $\frac{7}{8}$
- **c.** $\frac{2}{3}$
- **d.** $\frac{7}{24}$

Solve:

4. At a conference, the ratio of men to women was 2:3. How many men attended the conference of 270 people?
- **a.** 54
- **b.** 162
- **c.** 108
- **d.** 216

5. The team's ratio of wins to losses was 4:3. How many games did they win out of their 28 games?
a. 4
b. 7
c. 12
d. 16

6. Solve for a: $\frac{2}{5} = \frac{8}{5a}$
a. 2
b. 4
c. 6
d. 8

7. Solve for x: $\frac{x+2}{6} = \frac{9}{3}$
a. 16
b. 20
c. 24
d. 48

8. It takes 3 cups of sugar to make 90 cookies. How much sugar will you need for 60 cookies?
a. 6 cups
b. 5 cups
c. 4 cups
d. 2 cups

9. What percent of 48 is 36?
a. 75%
b. 70%
c. 80%
d. 60%

10. 15 is 30% of what number?
a. 40
b. 50
c. 60
d. 70

Review

Hour 14

Ratios and Proportions: Mixture and Work Problems

CHAPTER SUMMARY

In this hour you will learn about …

- Solving percent of increase or decrease problems
- Setting up and solving mixture problems
- Setting up and solving work problems

Ratios, proportions, and percents help you solve many different practical problems, from adjusting a recipe's ingredients to comparing the time you spend on the phone at work to the time you spend in meetings. In this next hour, you will learn additional ways percents are useful when analyzing change, whether it's the change in the price of an item that's now on sale or the change in your utility bills from one month to the next. You will also be able to calculate not only the amount of different items needed to make a certain mixture, but also the amount of time it will take to get a job done if others help you. Algebra is the key to all these real-life problems, even though they are very different situations.

Solving Percent of Increase or Decrease Problems

The $\frac{\text{is}}{\text{of}} = \frac{\%}{100}$ proportion gives you an efficient way to work with many percent problems. But it doesn't apply in some situations where percents often occur, such as when you are comparing prices, enrollment numbers, business expenses, or other statistics and want to know the percent of change.

Consider the following problem:

Your cable TV bill is now $90 per month. Last year you lived in another city and paid only $72 for the same service. What is the percent of increase for your cable service?

Solution: This percent problem is about change, and even though you can use the proportion from Hour 13 for many percent calculations, the following proportion helps you find the percent of change from an original amount to a new amount:

$$\frac{\text{change in price}}{\text{original price}} = \frac{\%\text{ of change}}{100}$$

In the cable TV example here, you know the original price, $72, and the new price, $90. The proportion uses the original price, but it also uses the change in price—not the new price. Subtraction gives you the change in price:

$$90 - 72 = 18$$

You can now put all the numbers into the proportion, using x for the unknown % of change:

$$\frac{\text{change in price}}{\text{original price}} = \frac{\%\text{ of change}}{100}$$

$$\frac{18}{72} = \frac{x}{100}$$

The cross-product process helps you create the next equation:

$$18 \cdot 100 = 72 \cdot x$$
$$1{,}800 = 72x$$

Dividing both sides by 72:

$$25 = x$$

Your cable bill has increased 25%.

This proportion—$\frac{\text{change in price}}{\text{original price}} = \frac{\%\text{ of change}}{100}$—works every time you want to calculate percent of change, whether it is an increase or a decrease.

If you always find the change in price by using "new price" minus "old price," the percent answer will be positive if it was an increase, and it will be negative if it was a decrease, as shown in the next example:

Your employer had to cut back everyone's hours this month to save on expenses. You now work 20 hours a week, and you were working 25 hours each week. What is the percent of change in your hours per week?

Solution: This problem is about work schedule change, not price change, but the change proportion still works. It can be shortened so that it applies to all percent-of-change problems:

$$\frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

Put your numbers in the proportion. Use 25 for "original" because you were working 25 hours a week before the slowdown. What is "change"? Remember to use "new" minus "old" to find "change." Your "new" schedule is 20 hours, and your "old" was 25.

$$\text{new} - \text{old} \rightarrow 20 - 25 = 20 + (-25) = -5$$

The numbers can now be put into the proportion:

$$\frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

$$\frac{-5}{25} = \frac{x}{100}$$

The cross-product creates the equation:

$$-5 \cdot 100 = 25 \cdot x$$
$$-500 = 25x$$
$$-500 \div 25 = 25x \div 25$$
$$-20 = x$$

Your hours per week have decreased 20% because a negative percent shows a decrease, just as a positive percent shows an increase in your cable bill.

Let's look at a utility bill comparison:

Last month your electricity bill was $68. The month before that it was $80. What is the percent of change?

Solution: Find the change by subtracting "new" minus "old." Careful with this one—the words make it a bit difficult to know which bill is new and which is old. Your "new" bill is $68, and your "old" bill was $80:

$$\text{new} - \text{old} \rightarrow 68 - 80 = -12$$

Put the numbers into the proportion:

$$\frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

$$\frac{-12}{80} = \frac{x}{100}$$

The cross-product gives the following equation:

$$-12 \cdot 100 = 80 \cdot x$$
$$-1{,}200 = 80x$$
$$-1{,}200 \div 80 = 80x \div 80$$
$$-15 = x$$

Your utility bill decreased 15%, from \$80 to \$62.

The two proportions for all percent problems

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100} \text{ and } \frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

are very easy to use, and you should memorize them. It helps that they both have the $\frac{\%}{100}$ after the equals signs. You will need to remember the is/of and the change/original ratios before the equals signs, but writing both proportions a few times will help keep them in your memory.

The percent-of-change proportion can lead to wrong answers if you forget that the first denominator must be the original number. Many people mistakenly use the new number. If you remember to always use the original, this proportion will be invaluable in finding percent of change, as well as finding other terms of the proportion if you already know the percent of change, as in the following example:

Enrollment at a local college increased 20% this year compared to a year ago. There are 640 more students this year than last. How many students were enrolled last year?

Solution: Use the percent-of-change proportion.

$$\frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

You already know the "change" is 640, the percent is 20, and x is the unknown percent, so these terms can be put into the proportion:

$$\frac{\text{change}}{\text{original}} = \frac{\%}{100}$$

$$\frac{640}{x} = \frac{20}{100}$$

The cross-product creates the equation:

$$64{,}000 = 20x$$
$$64{,}000 \div 20 = 20x \div 20$$
$$3{,}200 = x$$

There were 3,200 students enrolled last year.

Now try these percent of change problems, checking your answers on page 174.

1. The monthly rent for your apartment went up from $560 to $644. What is the percent of increase?
2. A pair of shoes you bought regularly costs $48, but today it's on sale for only $36. What is the percent of change?
3. You saved $15 when you bought a sweater today at a store that was having a sale of 25% off regular price. What was the regular price of the sweater?

Proportions have played a large role in the percent work so far this hour. The mixture problems that follow show that algebra comes in handy in lots of places, from labs to grocery stores.

Setting Up and Solving Mixture Problems

When different items are combined to make a mixture, whether it's a saline solution or granola cereal, percents are often an important part of finding the equation you need to solve.

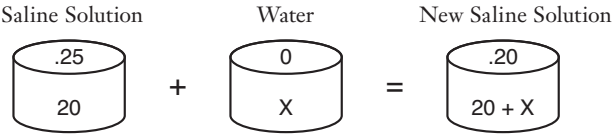
Let's look first at a saline solution problem. A chemist in a lab has 20 liters of a 25% saline solution. How many liters of water should be added to make a solution that is 20% saline?

Drawing the liters of solution as barrels is helpful. In each mixture problem, the percents are changed to decimal numbers. The lid of each barrel shows the percent of that barrel's salinity, written in decimal form. The lower part of each barrel shows the number of liters in that barrel.

JUST A MINUTE

Percents can always be written in decimal form by moving the decimal point two places to the left: 85% = .85; 5% = .05; 800% = 8. Decimal numbers can always be written as percents by moving the decimal point two places to the right: .18 = 18%; .01 = 1%; 2 = 200%.

In this problem, the 25% saline solution (in the first barrel, with .25 on the lid) will have water (in the second barrel) added to make a new barrel. Water is 0% saline, so 0 is on the lid of the second barrel. The new barrel needs to be 20% saline, so .20 is on the lid. The chemist has 20 liters in the first barrel and will be adding x liters in the second barrel to make a new barrel that will hold $20 + x$ liters. The barrels are shown here:



This illustration lets you see the problem clearly, and the equation is easily written from the barrels. You multiply the numbers in each barrel because "of" always indicates multiplication:

$$.25 \cdot 20 + 0 \cdot x = .20 \cdot (20 + x)$$
$$5 = .20 \cdot 20 + .20 \cdot x$$
$$5 = 4 + .2x$$
$$5 - 4 = 4 + .2x - 4$$
$$1 = .2x$$
$$1 \div .2 = .2x \div .2$$
$$5 = x$$

JUST A MINUTE

Decimal numbers that end in a zero to the right of the decimal point can be written with or without the zero(s): .50 = .5; 3.00 = 3.

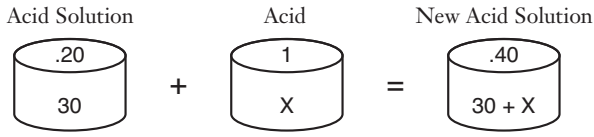
5 liters of water need to be added to make a 20% saline solution.

The following mixture problem shows how to increase the acidity in a solution. The method is similar to the saline problem, and the barrels will help you write the equation.

A chemist in a lab has 30 liters of a 20% acidic solution. How many liters of acid should be added to make a solution that is 40% acidic?

The 20% acid solution (in the first barrel, with .20 on the lid) will have acid (in the second barrel) added to make a new barrel. Acid is 100% acidic, and 100% = 1 in decimal form, so 1 is on the lid of the second

barrel. The new barrel needs to be 40% acid, so .40 is on the lid. The chemist has 30 liters in the first barrel and will be adding x liters in the second barrel to make a new barrel that will hold $30 + x$ liters. The barrels are shown here:



The equation can now be written from the barrels' labels:

$$.20 \cdot 30 + 1 \cdot x = .40 \cdot (30 + x)$$

JUST A MINUTE

When an equation has a variable on both sides of the equals sign, move the variable with the smaller coefficient: $2x + 5 = 5x - 7 \rightarrow 2x + 5 - 2x = 5x - 7 - 2x \rightarrow 5 = 3x - 7 \rightarrow 5 + 7 = 3x - 7 + 7 \rightarrow 12 = 3x \rightarrow x = 4$.

$$\begin{aligned}
6 + x &= .40 \cdot 30 + .40 \cdot x \\
6 + x &= 12 + .4x \\
6 + x - .4x &= 12 + .4x - .4x \\
6 + .6x &= 12 \\
6 + .6x - 6 &= 12 - 6 \\
.6x &= 6 \\
.6x \div .6 &= 6 \div .6 \\
x &= 10
\end{aligned}$$

Adding 10 liters of acid will give a new solution that is 40% acidic.

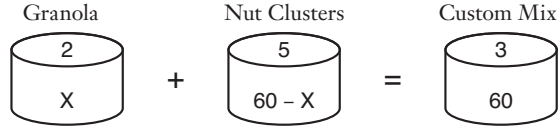
The barrel system is an efficient way to organize the information in a rather complex real-life situation to be used in an equation. When percents are involved, they are changed to decimals. In the next example, dollars are the units, along with pounds instead of liters, but the barrels help you see that it's really just an algebra problem that is easily solved.

Consider a granola mixture problem:

A store's produce department sells granola in bulk and makes its own custom mixture from a combination of granola and nut clusters. The granola costs \$2 per pound, and the nut clusters cost \$5 per pound. They

want to make 60 pounds of the custom mixture that will cost $3 per pound. How much of the granola and nut clusters will it take?

Solution: The first barrel is for the granola, and the second barrel is for nut clusters. The new mixture is in the barrel after the equals sign. This problem doesn't give an amount for either of the ingredients, but you do know there will be a total of 60 pounds in the new mixture barrel. If you let x be the pounds of granola, then $60 - x$ is the pounds of nut clusters. (Adding x and $60 - x$ equals 60, which is the total you want in the new barrel.) The barrels can be labeled as shown here:



Write the equation from the labels on the barrels:

$$2 \cdot x + 5 \cdot (60 - x) = 3 \cdot 60$$
$$2x + 5 \cdot 60 - 5 \cdot x = 180$$
$$2x + 300 - 5x = 180$$
$$-3x + 300 = 180$$
$$-3x + 300 - 300 = 180 - 300$$
$$-3x = -120$$
$$-3x \div -3 = -120 \div -3$$
$$x = 40$$

Because x is the pounds of granola, it will take 40 pounds of granola and 20 pounds (from $60 - x$) of the nut clusters to make 60 pounds of the custom mixture.

Now try the following mixture problems, checking your answers on pages 174–175.

4. How many liters of water should be added to 10 liters of 40% acidic solution to make a new solution that is 25% acidic?
5. You want to make a new candy mixture from jelly beans and chocolate candies. The jelly beans cost $2 a pound, and the chocolate costs $5 a pound. How much of each candy will you need to make 30 pounds of the new mixture that costs $4 a pound?

Mixture problems can most efficiently be shown as barrel drawings that lead you directly to the algebraic equation. Work problems also have a specific method that helps you write the equation. Let's finish this hour learning about the effect of people working together to get a job done.

Setting Up and Solving Work Problems

Setting up work schedules for a large business or a small shop has many elements that need to be considered. Algebraic equations are a big part of the decision making that goes into analyzing how long it takes more than one person to do a certain job.

In this last part of Hour 14, you may be surprised when you see how long it takes two people to do a job that they usually do separately.

Consider the following problem:

If it takes you 20 minutes to load packages onto the delivery truck and it takes Joe 30 minutes to do the same loading job, how long will it take if you both work together?

Solution: The method used to solve this type of problem is based on work rates. The following general work equation helps you find the time (t) it takes for two people—a and b—to do a job when they work together.

$$\frac{t}{a} + \frac{t}{b} = 1$$

The variables a and b represent the time it takes each person to do the given job when they are working alone. Putting in the example's numbers creates the following equation:

$$\frac{t}{a} + \frac{t}{b} = 1$$

$$\frac{t}{20} + \frac{t}{30} = 1$$

PROCEED WITH CAUTION

Recall that a proportion is two ratios that are equal to each other. The equations in these work-related problems are not proportions: the two ratios are not equal to each other—they are added to equal 1.

To solve for t, you need to move the 20 and the 30 to the other side of the equals sign. This problem isn't a proportion, so the cross-product method won't help solve this equation.

The key to isolating the variable t is the multiplication property of equality, which happens to be the same property that helps you solve proportions. If you multiply both sides of the equation by the least common denominator (LCD) for 20 and 30, the equation will be in a much simpler form.

30 is not the LCD because 20 is not a factor of 30. The easiest way to find the LCD for 20 and 30 is to multiply the larger number, 30, by 2. This equals 60, and 20 is a factor of 60, so the LCD is 60. The solution steps for the equation are shown here:

$$\frac{t}{20} + \frac{t}{30} = 1$$

$$60 \cdot \left(\frac{t}{20} + \frac{t}{30} \right) = 1 \cdot 60$$

$$60 \cdot \frac{t}{20} + 60 \cdot \frac{t}{30} = 60$$

$$3t + 2t = 60$$

$$5t = 60$$

$$5t \div 5 = 60 \div 5$$

$$t = 12$$

Working together, it will take only 12 minutes for you and Joe to load the delivery truck, as long as you both work at your regular speeds.

The answer to this type of work-together problem is usually a fraction, as in the following example:

You work for the maintenance department of a large apartment building. Before new tenants move in, you paint the walls of each apartment. If you can paint all bedroom walls and ceiling in a two-bedroom apartment in three hours and another employee can paint the same number of walls and the ceiling in the same apartment in four hours, how long will it take if you both work together to paint one apartment?

Solution: The general work equation is used for this problem:

$$\frac{t}{a} + \frac{t}{b} = 1$$

$$\frac{t}{3} + \frac{t}{4} = 1$$

To find the LCD, multiply 4 times 2, which is 8. Since 3 is not a factor of 8, multiply 4 times 3, which is 12 and the LCD. Multiplying both sides of the equation by 12 will help you isolate the variable:

$$\frac{t}{3} + \frac{t}{4} = 1$$

$$12 \cdot \left(\frac{t}{3} + \frac{t}{4}\right) = 1 \cdot 12$$

$$12 \cdot \frac{t}{3} + 12 \cdot \frac{t}{4} = 12$$

$$4t + 3t = 12$$

$$7t = 12$$

$$7t \div 7 = 12 \div 7$$

$$t = \frac{12}{7}$$

Using a calculator to divide 12 by 7 gives a decimal answer of about 1.7. The times in the problem are in hours, as is the answer, so it will take about 1.7 hours if you both paint the apartment together, or a little less than 2 hours.

The work equation can also be used if three or more people are all working together—the LCD may be a little harder to find, but it can be done. The equation for three workers would be as follows:

$$\frac{t}{a} + \frac{t}{b} + \frac{t}{c} = 1$$

This problem-solving technique applies only to a job that can be done by more than one person at the same time, not an assembly-line type job where many lines of work are in progress. The ability to calculate the amount of time it takes for people to do a job working together instead of separately relies heavily on algebraic skills, and you have mastered them.

Now try these work problems, checking your answers on page 175.

6. When you work alone, it takes you 2 hours to mow the lawn on a rider mower. Your brother can mow it in 4 hours walking behind a self-propelled mower. How long will it take if you both work together?

7. You can vacuum the restaurant's floor area after closing time in 10 minutes. Using an older vacuum cleaner that doesn't work as well, your co-worker takes 15 minutes. If you both work together to get the job done, how long will it take?

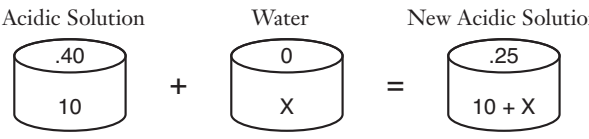
Answers to Sample Problems

1. $\frac{\text{change}}{\text{original}} = \frac{\%}{100} \rightarrow \frac{84}{560} = \frac{x}{100} \rightarrow 8400 = 560x \rightarrow 84008 \div 560 = 560x \div 560 \rightarrow 15 = x$. The percent of increase is 15%.

2. $\frac{\text{change}}{\text{original}} = \frac{\%}{100} \rightarrow \frac{-12}{48} = \frac{x}{100} \rightarrow -1200 = 48x \rightarrow -1200 \div 48 = 48x \div 48 \rightarrow -25 = x$. The percent of decrease is 25% (–25% is a decrease).

3. $\frac{\text{change}}{\text{original}} = \frac{\%}{100} \rightarrow \frac{15}{x} = \frac{25}{100} \rightarrow 1500 = 25x \rightarrow 1500 \div 25 = 25x \div 25 \rightarrow 60 = x$. The regular price of the sweater was $60.

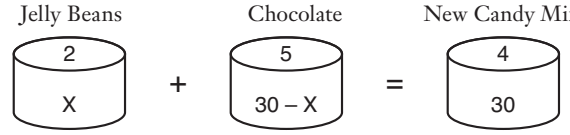
4.



$.40 \cdot 10 + 0 \cdot x = .25 \cdot (10 + x)$
$4 = .25 \cdot 10 + .25 \cdot x$
$4 = 2.5 + .25x$
$4 - 2.5 = 2.5 + .25x - 2.5$
$1.5 = .25x$
$1.5 \div .25 = .25x \div .25$
$6 = x$

6 liters of water need to be added.

5.



$2 \cdot x + 5 \cdot (30 - x) = 4 \cdot 30$
$2x + 5 \cdot 30 - 5 \cdot x = 120$
$2x + 150 - 5x = 120$
$-3x + 150 = 120$
$-3x + 150 - 150 = 120 - 150$
$-3x = -30$
$-3x \div -3 = -30 \div -3$
$x = 10$

10 pounds of jelly beans and 20 pounds of chocolate

6. $$\frac{t}{2} + \frac{t}{4} = 1$$
$$8 \cdot \left(\frac{t}{2} + \frac{t}{4}\right) = 1 \cdot 8$$
$$8 \cdot \frac{t}{2} + 8 \cdot \frac{t}{4} = 8$$
$$4t + 2t = 8$$
$$6t = 8$$
$$6t \div 6 = 8 \div 6$$
$$t = \frac{8}{6} = \frac{4}{3}$$

It will take about 1.3 hours to mow the lawn if you both work together.

7. $$\frac{t}{10} + \frac{t}{15} = 1$$
$$30 \cdot \left(\frac{t}{10} + \frac{t}{15}\right) = 1 \cdot 30$$
$$30 \cdot \frac{t}{10} + 30 \cdot \frac{t}{15} = 30$$
$$3t + 2t = 30$$
$$5t = 30$$
$$5t \div 5 = 30 \div 5$$
$$t = 6$$

It will take 6 minutes if you both work together.

Review

Hour's Up!

Based on what you've learned in Hour 14, do the following problems. Check your answers with the solution key in Appendix B.

1. Your water bill last month was $24. The month before that, it was $20. What is the percent of change?
   a. Increase of 10%
   b. Increase of 20%
   c. Decrease of 16%
   d. Decrease of 20%

2. The monthly rent for your new apartment is $605. You paid $550 each month for your old apartment. What is the percent of change?
   a. Increase of 9%
   b. Decrease of 9%
   c. Decrease of 10%
   d. Increase of 10%

3. You bought a pair of boots on sale for $90. The regular price was $120. What was the percent of change?
   a. Decrease of 25%
   b. Increase of 25%
   c. Decrease of 33%
   d. Increase of 33%

4. Your boss needs you to work more hours next week to help cover another employee who will be on vacation. This week you are working 25 hours, and next week you are scheduled for 30 hours. What is the percent of change?
   a. Increase of 17%
   b. Decrease of 17%
   c. Increase of 20%
   d. Decrease of 20%

Review

5. How much acid should be added to 20 liters of 25% acidic solution to make a new solution that is 40% acidic?
   a. 5 liters
   b. 10 liters
   c. 15 liters
   d. 20 liters

6. How much water should be added to 50 liters of 30% saline solution to make a new solution that is 20% saline?
   a. 10 liters
   b. 15 liters
   c. 20 liters
   d. 25 liters

7. Pecans and walnuts will be combined to make 40 pounds of nut mixture that costs $6 per pound. The pecans cost $8 per pound, and the walnuts cost $3 per pound. How many pounds of pecans and of walnuts will it take to make the nut mixture?
   a. 24 pounds of pecans, 16 pounds of walnuts
   b. 16 pounds of pecans, 24 pounds of walnuts
   c. 20 pounds of pecans, 20 pounds of walnuts
   d. 10 pounds of pecans, 30 pounds of walnuts

8. You can load 600 boxes into the semi-trailer in 30 minutes. Your co-worker is very fast and takes only 15 minutes to do the same job. How long will it take if you both load the truck together?
   a. 5 minutes
   b. 10 minutes
   c. 12 minutes
   d. 15 minutes

9. It takes you 40 minutes to wash your car by hand in the driveway. Your brother washes it in 30 minutes. How long will it take if you both wash it together?
   a. About 10 minutes
   b. About 13 minutes
   c. About 17 minutes
   d. About 20 minutes

Review

10. You can unload and stack a cord of wood in 2 hours. Your brother takes 3 hours to do the same job. How long will it take if you both work together?

a. 1.2 hours
b. 2.2 hours
c. 2.4 hours
d. 2.6 hours

Part V

Systems of Linear Equations

Hour 15 Equations with Two Variables: Graphing

Hour 16 Equations with Two Variables: Equations of Lines

Hour 17 Solving Systems of Equations: Three Methods

Hour 18 Solving Systems of Equations: Combined Operations

Hour 19 Solving Inequalities: Graphing

Hour 20 Solving Inequalities: Absolute Value

Hour 15

Equations with Two Variables: Graphing

Chapter Summary

In this hour you will learn about …

- Plotting ordered pairs on a graph
- Graphing two-variable equations using the intercepts
- Finding the slope of a line, given two points
- Writing the slope-intercept form of a linear equation

The real-life applications of ratios, proportions, and percents in the previous two hours of study brought together your knowledge of polynomials, algebraic fractions, and simplifying expressions. The variety of real-world problems you can now solve is a perfect example of why it's important to understand and use algebra with confidence.

Think back for a moment to solving equations. Your goal in solving all the equations so far has been to isolate the variable. Each equation had a single variable, although sometimes it appeared on both sides of the equals sign. This hour marks a turning point in your study of algebra. Most of the work from now on focuses on working with two different variables in each equation, which will expand your thinking skills greatly!

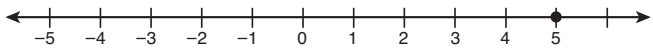
Plotting Ordered Pairs on a Graph

The number line lets you graph points anywhere along its length, whether the number is positive, negative, zero, a whole number, a fraction, or a decimal—each number has its place somewhere on the number line. An equation with one variable might have one solution, or it might have no solution, as in the following examples.

Solve: $x - 2 = 3$

Solution: $x - 2 + 2 = 3 + 2 \rightarrow x = 5$

Graph the solution on a number line:



Solve: $x + 4 = x + 3$

Solution: $x + 4 = x + 3 \rightarrow x + 4 - x = x + 3 - x \rightarrow 4 = 3$

Because 4 cannot equal 3, this equation has no solution.

An equation with two different variables might have many solutions, as in the next example.

Solve: $x + y = 6$

Solution: With two variables, x and y can be many possible answers, including $x = 1$ when $y = 5$. Instead of having a single answer, this equation has a pair of numbers as an answer. To graph this solution, you need a two-dimensional graph.

This new graph begins with the number line you have been using for plotting points. Through the zero point on that line, a vertical line is drawn. This new line is labeled with positive numbers above the zero and negative numbers below the zero.

The horizontal line is called the x-axis, and the vertical line is the y-axis. The point you are graphing has an x value of 1 and a y value of 5. The most efficient way to describe these values is as a pair of numbers:

$$(x,y) \rightarrow (1,5)$$

Another possible solution to the equation $x + y = 6$ is $x = 4$ when $y = 2$, and that pair of numbers would be written:

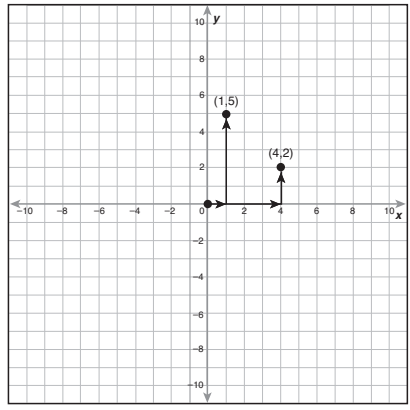
$$(x,y) \rightarrow (4,2)$$

To graph these two *ordered pairs*, begin with the first pair, (1,5). Because $x = 1$, find +1 on the horizontal line (the x-axis). Because $y = 5$, rise up 5 marks to +5. This point is shown on the following graph.

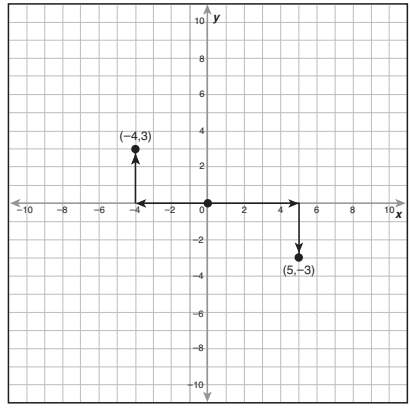
STRICTLY DEFINED

An **ordered pair** is a pair of numbers that represents a solution to an equation with two variables. It is called an "ordered" pair because the location of (3,4) is not the same as the location of (4,3).

The second pair, (4,2), is graphed similarly: because $x = 4$, find +4 on the x-axis, then rise up 2 marks to +2. These two points have been graphed in the following figure, with arrows showing the process:



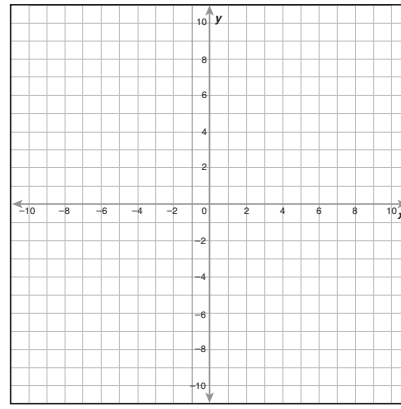
Ordered pairs can also have negative numbers as one or both of the values. To graph (5,–3), begin at +5 on the *x*-axis, then go down 3 marks to –3. If the ordered pair is (–4,3), begin at –4, to the left of 0 on the *x*-axis, then go up 3 to +3. This graphing process for each of these points follows:



FYI The point (0,0) is often called the origin because you start at (0,0) when you graph each ordered pair. An ordered pair can also have 0 as the *x* or *y* value: the point (0,4) is up 4 on the *y*-axis; the point (–3,0) is 3 to the left on the *x*-axis.

Now try graphing the following ordered pairs on the following graph, checking your answers on page 194.

1. (6,2)
2. (–2,–4)
3. (–5,5)

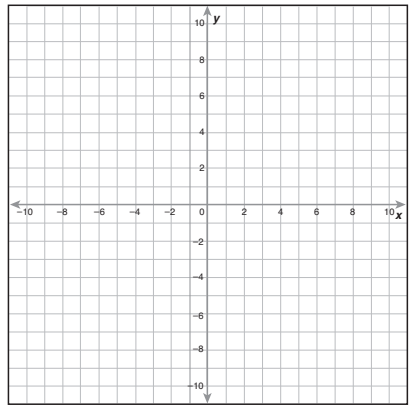


Graphing ordered pairs helps you find many of the solutions for equations with two variables, as you will learn in the next section.

Graphing Two-Variable Equations Using the Intercepts

As you saw in the earlier equation sample, when there are two different variables in an equation, there are many solutions, shown as ordered pairs.

Look again at the equation $x + y = 6$. Two of the ordered pair solutions you graphed on the axes were (1,5) and (4,2). Plot them again on the following graph:



With the edge of a ruler, index card, or any straightedge, draw the line to connect these two points, extending the line up past and down below both points, going as far as you can in both directions while still staying on the graph. Did your line go through any other whole-number points? If you graphed the points correctly and drew the line joining them carefully, the line should go through each of these points: (0,6), (2,4), (3,3), (5,1), and (6,0).

Putting each of these pairs of numbers into the equation $x + y = 6$ should make a true equation. For example, the ordered pair (0,6) would become $0 + 6 = 6$, which is true. In fact, each point's x and y values will sum to 6.

This tells you that if you can find just two ordered pairs that make an equation true, graphing those points and drawing the line that connects them will show you all the ordered pairs that make the equation true.

Return once again to that sample equation, $x + y = 6$. The two ordered pairs that are the easiest to find are when $x = 0$, so $y = 6$ and when $y = 0$, so $x = 6$. These ordered pairs are (0,6) and (6,0). Graphing those in the very beginning and drawing the line that connects them gives you all the other solutions that are listed above.

In fact, the points (0,6) and (6,0) are so helpful as you try to find the solutions to the equation that they have special names:

(0,6) is where the line meets the *y*-axis, so (0,6) is called the *y*-intercept.
(6,0) is where the line meets the *x*-axis, so (6,0) is called the *x*-intercept.

Together the points (0,6) and (6,0) are called the intercepts, and they are not only easy to find from the original equation, but they are easy to plot on the graph.

Look back at your graph of $x + y = 6$. The line should have also gone through other easily named points, including (–1,7) and (–2,8), as well as (7,–1) and (8,–2). Notice that each of these ordered pairs makes a true statement when they are put into the equation $x + y = 6$. They are all certainly solutions, just as the points listed previously are.

There is actually an *infinite* number of points on this line that are all solutions for the equation, and they include all the whole-number pairs such as (9,–3) and (10,–4). But don't stop there—all the fractional and decimal ordered pairs on this same line are also solutions, including (2.5,3.5), (2.51,3.49), and (–1.3,7.3).

STRICTLY DEFINED

An **infinite** number of points means that there are so many, they cannot be counted; the concept of an infinite number indicates that the number increases without an upper bound or is never-ending.

The list is endless, but the good news is that by drawing the line that connects any two of the easily found solutions, you are able to show all the solutions because the solutions are all the points on the line.

Let's look at another equation and graph its intercepts:

$$x - y = 2$$

Solution: Let x equal $0 \rightarrow 0 - y = 2$

To solve $0 - y = 2$, adding y to both sides makes the equation a little easier to work with:

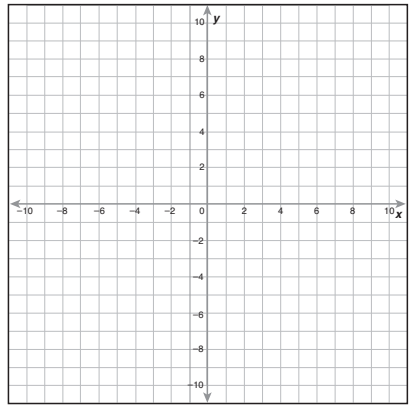
$$0 - y + y = 2 + y \rightarrow 0 = 2 + y \rightarrow 0 - 2 = 2 + y - 2 \rightarrow -2 = y$$

So (0,–2) is a solution for this equation, and it's an intercept on the graph.

Now let y equal $0 \rightarrow x - 0 = 2 \rightarrow x = 2$.

So (2,0) is a solution for this equation, and it's also an intercept on the graph.

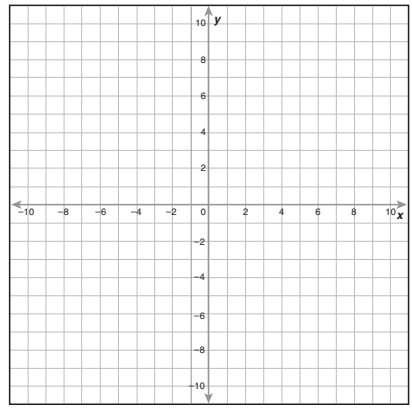
Plot these two key points on the following graph. Connecting them shows you the line containing every ordered pair that is a solution to this equation.

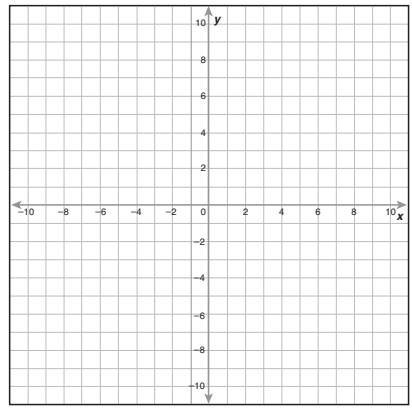


Your line should go through (6,4) and (–1,–3), as well as many others. Each of these ordered pairs makes a true statement when they are put into $x - y = 2$.

Now try to graph the intercepts of the following equations, draw the line, and list two other ordered pairs that are also solutions. Check your answers on pages 194–195.

4. $x + y = 5$
5. $x + y = -6$

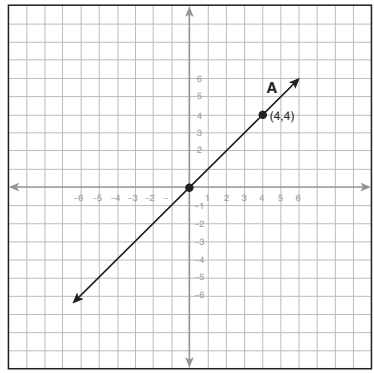


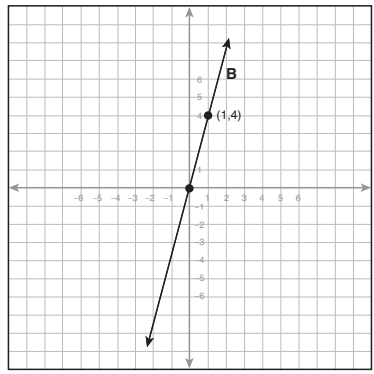


The graphs of the two equations in previous problems 4 and 5 resulted in lines that are not horizontal or vertical. Comparing the lines, you see that both slant downward to the right. Your skills with ratios will help you analyze these and other lines in the next section.

Finding the Slope of a Line, Given Two Points

Consider the lines in the two following graphs. Line A connects the points (0,0) and (4,4), and line B connects the points (0,0) and (1,4). Each line has been extended above and below the points a short distance.





As you read the following facts about lines A and B, refer to the previous graphs.

Both lines are slanted, and they both rise to the right, with the second line rising more steeply.

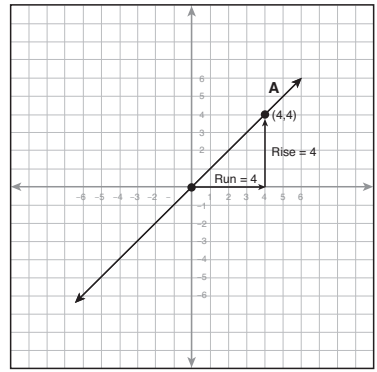
Both lines go through (0,0), with line A going through (4,4), while line B rises more steeply and goes through (1,4).

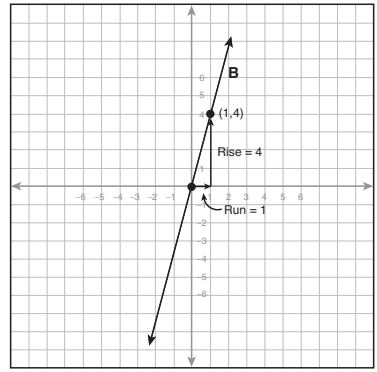
In algebra, the word that describes a line's slant is *slope*. The slope of a line is the ratio of a line's rise to its run. You'll use a formula to find this ratio, but let's look at two examples first:

Lines A and B have been drawn again on the following graphs, and this time the rise and run of each line are labeled.

STRICTLY DEFINED

A line's **slope** is the steepness of the line as you move from one point to another along the line. Slope is based on how much the line rises compared to its horizontal change, which is the ratio of rise to run.





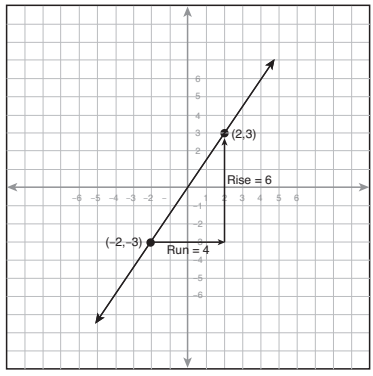
Line A's ratio of rise to run is 4:4, or $\frac{4}{4} = 1$.

Line B's ratio of rise to run is 4:1, or $\frac{4}{1} = 4$.

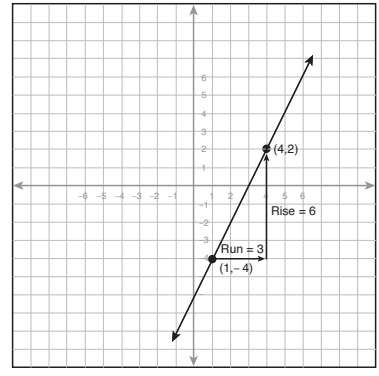
Line B is rising more steeply than line A. Line B's slope is 4, which is much higher than line A's slope of 1. This is consistent with the slopes of all graphed lines: the steeper the line, the higher the absolute value of the slope.

Now try to find the slope of the following lines that are graphed here, checking your answers on page 195.

6.



7.



When an equation is graphed, you have a visual image of the line's slope. In the previous graphs, the *rise* and *run* are labeled for you, so finding the ratio of rise to run is easy. Also, both lines A and B are rising to the right, and their slopes are positive, which is true for all lines that rise to the right.

STRICTLY DEFINED

The **rise** of a line is the difference of the *y* values of two ordered pairs on that line. The **run** of a line is the difference of the *x* values of two ordered pairs on the line.

Looking at the graphed line does allow you to find the rise and run by counting. But there's also a way to find the rise and run without even seeing the line—all you need to know are the two ordered pairs that are on the line.

The rise is found by subtracting the *y* values of the two points, and the run is found by subtracting the *x* values of the same two points.

Try this method with a line that you already know the slope of, such as the line that goes through (0,0) and (1,4):

The *y* values of the ordered pairs are 0 and 4, so the rise is 0 – 4= –4.

The *x* values of the ordered pairs are 0 and 1, so the run is 0 – 1 = –1.

Slope is the ratio of rise to run $\rightarrow -4$ to $-1 \rightarrow) \frac{-4}{-1} = 4$.

You know the slope is 4 from using the rise and run labeled on the graph. You found the same slope by calculating the rise and run with the subtraction method.

Knowing two points on a line enables you to find the slope of that line without having to graph the line, and the subtraction method is in formula form here:

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}}$$

PROCEED WITH CAUTION

Be careful to use the *y* values of the ordered pairs as the numerator and the *x* values in the denominator when finding the slope of a line: the ratio of 4 to 3 is not the same as the ratio of 3 to 4.

For example, if the points (2,3) and (6,8) are on the line, to find the slope of this line, find the rise and run as follows:

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{3-8}{2-6} = \frac{-5}{-4} = \frac{5}{4}$$

The slope of the line containing (2,3) and (6,8) is $\frac{5}{4}$.

What if you listed the same two points as (6,8) and (2,3)? Will the slope be the same?

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{8-3}{6-2} = \frac{5}{4}$$

Yes, the slope is the same. If you graphed these two ordered pairs and drew the line connecting them, there is only one line—it doesn't matter which point you happen to graph before the other, so of course the slope is the same number, no matter which point you may choose as the "starting point."

All the slopes in this section have been positive, but slopes can be negative. Lines that fall to the right have negative slopes, as in the following example:

Find the slope of the line containing the points (3,5) and (7,0):

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{5-0}{3-7} = \frac{5}{-4} = -\frac{5}{4}$$

Graphing these two points would show you that the line does fall to the right, but using the slope formula confirms it. The slope of the line containing the points (3,5) and (7,0) is $-\frac{5}{4}$.

The final example shows how to find the slope when one or more of the x or y values in the ordered pairs are negative:

Find the slope of the line containing the points (–2,6) and (–5,–3):

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{6-(-3)}{-2-(-5)} = \frac{6+3}{-2+5} = \frac{9}{3} = \frac{3}{1}$$

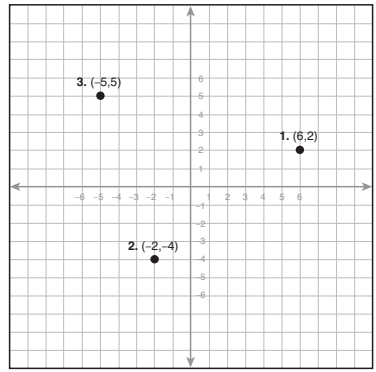
If you are careful with the subtraction process and watch the negative signs of the terms, your algebra skills will get you through this more complex-looking problem easily.

Now try to find the slope of the line containing the following points, checking your answers on page 195.

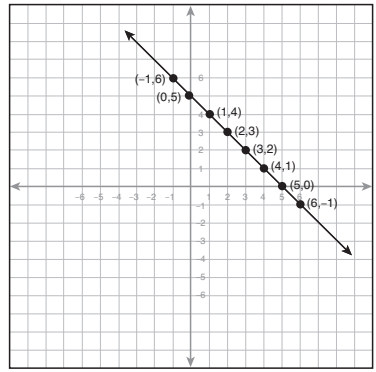
8. (0,8) and (10,0)
9. (–5,3) and (–2,–1)

Answers to Sample Problems

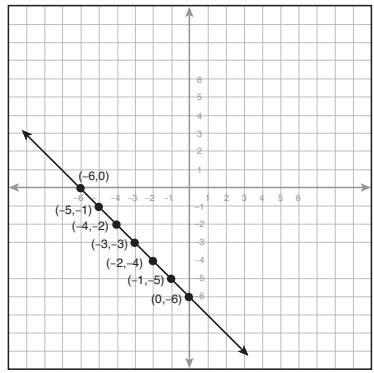
1., 2., 3.



4. $x + y = 5$



5. $x + y = -6$



6. The slope of the line is the ratio of rise to run. $\rightarrow \frac{6}{4} = \frac{3}{2}$

7. The slope of the line is the ratio of rise to run. $\rightarrow \frac{6}{3} = 2$

8. (0,8) and (10,0) $\rightarrow$ Slope =
$$\frac{\text{rise}}{\text{run}} = \frac{\text{difference of y values}}{\text{difference of x values}} = \frac{8 - 0}{0 - 10} = \frac{8}{-10} = -\frac{4}{5}$$

9. (–5,3) and (–2,–1) $\rightarrow$ Slope =
$$\frac{\text{rise}}{\text{run}} = \frac{\text{difference of y values}}{\text{difference of x values}} = \frac{3 - (-1)}{-5 - (-2)} = \frac{3 + 1}{-5 + 2} = \frac{4}{-3} = -\frac{4}{3}$$

Hour's Up!

Based on what you've learned in Hour 15, do the following problems. Check your answers with the solution key in Appendix B.

For problems 1 through 4, graph the intercepts of the following equation, and draw the line connecting the intercepts. Choose the ordered pair in each problem that is on the graphed line.

Review

Review

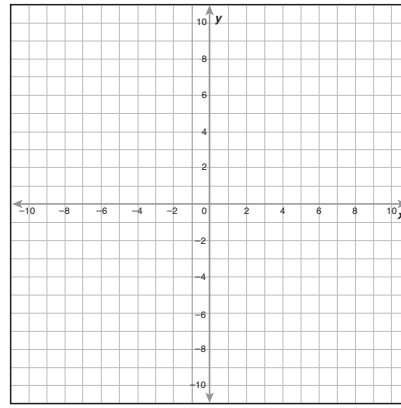
1. $x + y = -3$

a. (2,5)

b. (3,0)

c. (2,–5)

d. (1,–5)



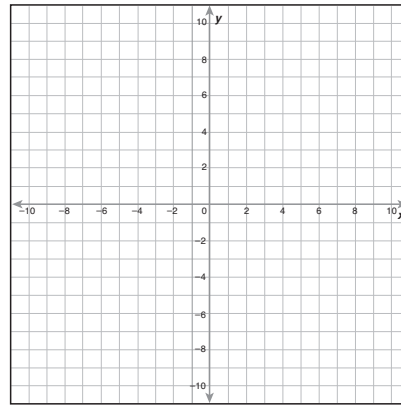
2. $x + y = 7$

a. (8,1)

b. (8,–1)

c. (1,8)

d. (1,–8)



Review

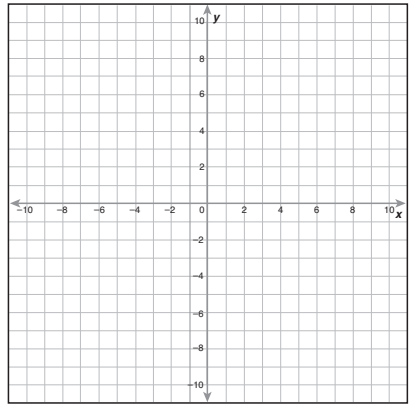
3. $2x + y = 4$

a. (2,–3)

b. (3,2)

c. (3,–2)

d. (2,3)



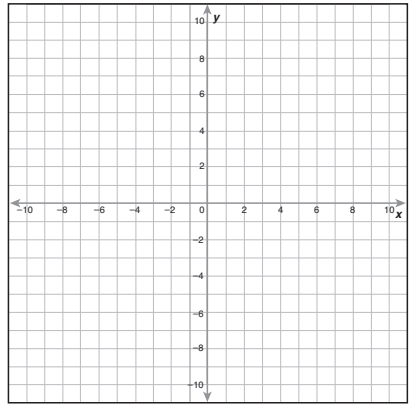
4. $x - y = 4$

a. (–1,–5)

b. (–5,–1)

c. (1,–5)

d. (0,4)



Review

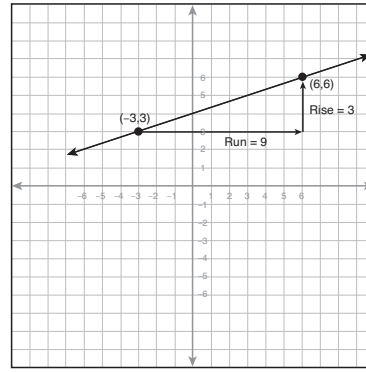
5. Find the slope of the line in the following graph:

a. 3

b. $\frac{1}{3}$

c. $-\frac{1}{3}$

d. –3



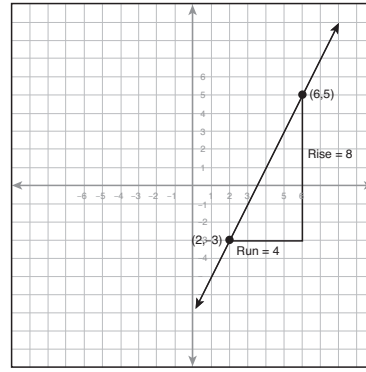
6. Find the slope of the line graphed next:

a. $-\frac{1}{2}$

b. $\frac{1}{2}$

c. 2

d. –2



7. Find the slope of the line containing the points (2,4) and (6,10).

a. $\frac{3}{2}$

b. $\frac{2}{3}$

c. $\frac{7}{6}$

d. $-\frac{3}{2}$

8. Find the slope of the line containing the points (–3,8) and (5,2).

a. $-\frac{4}{3}$

b. $\frac{4}{3}$

c. $\frac{3}{4}$

d. $-\frac{3}{4}$

9. Find the slope of the line containing the points (–1,–7) and (–3,7).

a. $\frac{1}{7}$

b. $-\frac{1}{7}$

c. –7

d. –1

10. Find the slope of the line containing the points (0,–4) and (6,–8).

a. $\frac{3}{2}$

b. $-\frac{3}{2}$

c. $\frac{2}{3}$

d. $-\frac{2}{3}$

Review

Hour 16

Equations with Two Variables: Equations of Lines

Chapter Summary

In this hour you will learn about …

- Finding the equation of a line from the slope and ***y***-intercept
- Finding the equation of a line from the slope and a point on the line
- Finding the equation of a line from two points on the line
- Solving direct variation problems
- Solving inverse variation problems

Last hour's graphing of points and lines from equations with an x and a y variable showed you one way to find many ordered pairs that are true for a given equation. Plotting the intercepts allowed you to draw the line that contains each equation's solutions.

This hour you will learn how to use a line's slope and key points to write that line's equation—in some ways, the reverse of the process you learned last hour—along with how to solve real-life problems about direct and inverse variation.

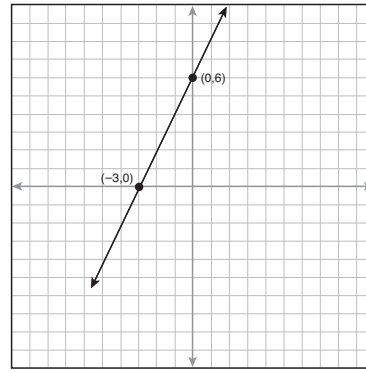
Finding the Equation of a Line from the Slope and *y*-Intercept

The graph of the equation $y - 2x = 6$ follows. When $x = 0$, $y = 6$, and when $y = 0$, $x = -3$. These two key points are the ordered pairs (0,6) and (–3,0). (0,6) is the y-intercept, and (-3,0) is the x-intercept.

JUST A MINUTE

Ordered pairs are always written as (*x*,*y*) points and are graphed with the first number as ***x*** and the second number as ***y***. The ordered pair (2,3) is graphed by moving two units to the right of (0,0) and then moving three units up.

The slope of this line is the ratio of rise to run → 6 to 3 → 2.



The preceding graph came from the equation $y - 2x = 6$.

If you move some of the terms and isolate the y variable, the equation would be this:

$$y - 2x = 6$$
$$y - 2x + 2x = 6 + 2x$$
$$y = 6 + 2x$$
$$y = 2x + 6$$

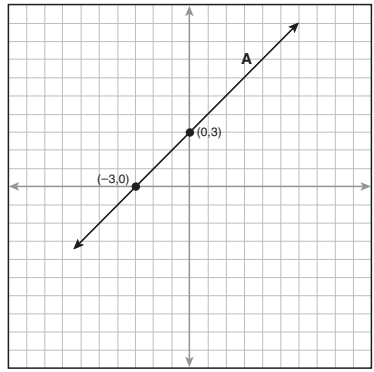
JUST A MINUTE

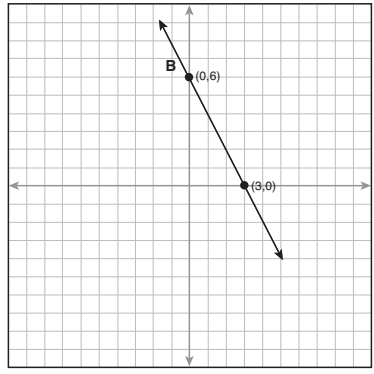
To "isolate the variable *y*" is the same as to "solve the equation for *y*." The *y* variable needs to be alone on the left side, and the *x* variable and the number, or constant, term need to be on the right side of the equals sign.

When the equation is written as $y = 2x + 6$, the 2 is the slope and the 6 is the y-intercept of the graph.

Looking at the intercepts and slope of the next two graphs might help you see this is not a coincidence—there's definitely a close connection between the numbers in the equation and the slope and y-intercept:

line A: $y = x + 3$ line B: $y = -2x + 6$





y-intercept: 3

$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3}{3} = 1$

Line A's equation:

$y = x + 3$

y-intercept: 6

$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{6}{-3} = -2$

Line B's equation:

$y = -2x + 6$

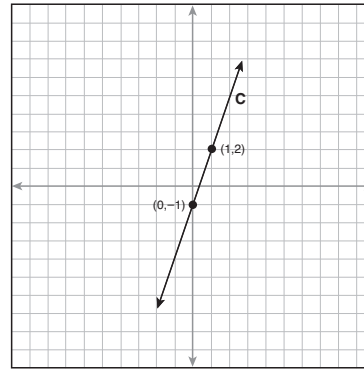
Did you notice that the slope and the *y*-intercept from each graph also appear in the equation? (The *x*-intercept is helpful for graphing, but it isn't included in the equation.)

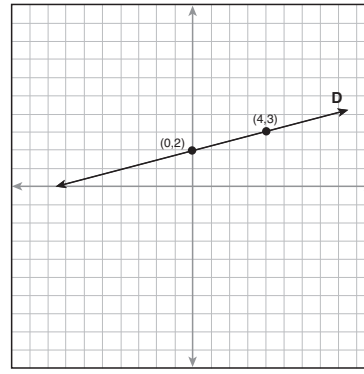
In each equation, the y is isolated on the left side of the equals sign. The x with its coefficient is on the right side, followed by a number, called a constant. The slope is the coefficient of the x term, and the y-intercept is the constant. This leads to the slope/intercept formula for the graph of a line:

> $y = mx + b$, where m equals the slope of the line and b equals the y-intercept

The following graphs give additional examples of possible m and b values, as shown by the slope and y-intercepts of these lines:

line C: $y = 3x - 1$ line D: $y = \frac{1}{4}x + 2$





FYI The key information you need from each graph is the slope of the line and the *y*-intercept—the *x*-intercept cannot be directly read from the equation of a line.

From the equation for line C, $y = 3x - 1$, you can see that $m = 3$ and $b = -1$. The graph of line C does have a *y*-intercept of –1 and a slope of 3.

From the equation for line D, $y = \frac{1}{4}x + 2$, you can see that $m = \frac{1}{4}$ and $b = 2$. The graph of line D does have a *y*-intercept of 2 and a slope of $\frac{1}{4}$.

The important part of this connection between the line and the equation is that if you know the slope and *y*-intercept for any line, you can write the equation for that line.

You might know the line's slope and *y*-intercept from examining the graph of the line, or you might simply be told what the slope and *y*-intercept are. Either way, the equation for the line is based on $y = mx + b$.

For example, the equation for the line with slope of 4 and *y*-intercept of –3 is:

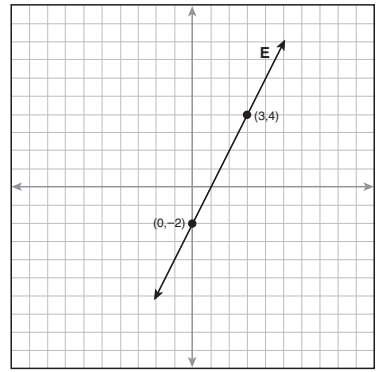
$$y = 4x - 3$$

And the equation for the line with slope of $-\frac{2}{3}$ and *y*-intercept of $\frac{4}{3}$ is:

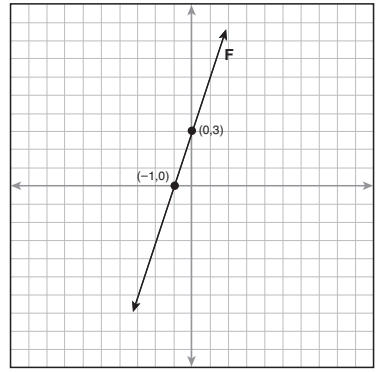
$$y = -\frac{2}{3}x + \frac{4}{3}$$

(This line would be rather difficult to graph, but it could be done.)

Now try to write the equation based on $y = mx + b$ for each of the following lines, using their slope and *y*-intercept for m and b. Check your answers on pages 216–217.

1. 

Equation of line E:

2. 

Equation of line F:

3. The slope of line G is –5, and the *y*-intercept is 8. Write the equation of line G.

The equation of a line can easily be written when you can find the slope and *y*-intercept from the graph or when you are told the values of the slope and *y*-intercept. Your algebra skills can also help you write the equation when the information you have is not quite so complete.

Finding the Equation of a Line from the Slope and a Point on the Line

If you know a line's slope (m) and y-intercept (b), you can write its equation with the help of $y = mx + b$. However, if all you know is the slope of the line and a random point on that line, how do you know the b value and the y-intercept?

You have at least two choices:

1. Graph the line, using the point and the line's slope.
2. Let algebra help you find the b value.

In the first choice, graphing is certainly possible, but it depends on your abilities with graph paper, straightedges, and so on. This could be challenging if the slope is a fraction or if the given point is something like (–8,28).

The second choice—using algebra to help you find the b value—makes perfect sense because you have been spending many hours learning algebra, and now it can be put to a good use. Let's try it.

Consider the following information about a line:

> The slope of the line is 3 and a point on the line is (4,6). What is its equation?
> You know that your goal is to find the values of m and b for this line and then put these values into $y = mx + b$. You already know that $m = 3$.
> You also know what x and y are: if the ordered pair (4,6) is on the line, then $x = 4$ and $y = 6$ in the equation $y = mx + b$. The process for finding the equation of this line follows:
> Begin with $y = mx + b$.

Of these variables, you know from the point (4,6) that x is 4 and y is 6. You are also told that the slope is 3, so m is 3. Put these values in for y, m, and x:

$$y = mx + b$$
$$6 = 3 \cdot 4 + b$$

Solve this equation for *b:*

$$6 = 3 \cdot 4 + b$$
$$6 = 12 + b$$
$$6 - 12 = 12 + b - 12$$
$$-6 = b$$

You now know that $b = -6$. Put $b = -6$ along with $m = 3$ into $y = mx + b$, and you have the equation you are looking for:

$$y = 3x - 6$$

TIME SAVER

Graphing your equation is one way to check your work, but another, more efficient way, is to put the given point's *x* and *y* values into your equation. Does 6 equal $3 \cdot 4 - 6$? Yes, so your equation is correct.

You could verify this by graphing this equation, and you would see that a line with *y*-intercept of –6 and slope of 3 does contain the point (4,6), but this verification isn't necessary. Your algebra skills will serve you well in this and many other equation-writing problems.

Follow the solution steps on the next example. Find the equation for the line that goes through the point (1,4) and has a slope of –2.

In this problem, the ordered pair (1,4) tells you that $x = 1$ and $y = 4$. The slope is –2, so $m = -2$. You need to find the b value. Put the values you know into $y = mx + b$*:*

$$4 = -2 \cdot 1 + b$$
$$4 = -2 + b$$
$$4 + 2 = -2 + b + 2$$
$$6 = b$$

Because $m = -2$ and $b = 6$, the equation for the line is $y = -2x + 6$.

Now try to write the equation for each of the following lines, checking your answers on page 217.

4. Find the equation of the line that has a slope of –8 contains the point (–3,2).
5. Find the equation of the line that contains the point (5,–6) and has a slope of –3.

When you know the point on a line and the line's slope, substituting the values for x, y, and m into $y = mx + b$ gives you a way to know what b equals, and that leads you directly to knowing the equation of the line.

You can now write the equation of a line from two different methods: analyzing its graph to find its slope and its y-intercept or knowing a point on the line and the line's slope. Depending on what you know about any given line, one of these methods can work. However, what if you only know two points that are on a line? Again, your algebra skills will help you find the equation.

Finding the Equation of a Line from Two Points on the Line

Consider the line that goes through the points (1,2) and (3,6). Can you find the equation that describes this line?

You know the slope-intercept formula is $y = mx + b$, where (x,y) is a point on the line, m is the slope of the line, and b is the y-intercept of the line. But knowing two of the points on the line doesn't really seem that helpful.

JUST A MINUTE

Recall that the slope—the ratio of rise to run—of a line can be found by subtracting the *y* values of the two ordered pairs for the rise and subtracting the *x* values for the run.

However, you can find the slope by using the rise-to-run ratio from Hour 15. With the two given points—(1,2) and (3,6)—the rise is $2 - 6 = -4$, and the run is $1 - 3 = -2$:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-4}{-2} = 2 \rightarrow m = 2 \text{ in } y = mx + b$$

Because you are given two of the points on this line, choose one point and put its x and y values into $y = mx + b$. Let's choose the point (1,2). The equation that can be solved to find b follows:

$$\begin{aligned} y &= mx + b \\ 2 &= 2 \cdot 1 + b \\ 2 &= 2 + b \\ 2 - 2 &= 2 + b - 2 \\ 0 &= b \end{aligned}$$

With $m = 2$ and $b = 0$, the equation for the line containing the points (1,2) and (3,6) is $y = 2x + 0$ or simply $y = 2x$.

JUST A MINUTE

You can use either of the two given points when substituting the *x* and *y* values into $y = mx + b$. Any line has only one slope, and both points will lead you to the same *b* value (*y*-intercept) for the line.

What if you had chosen the point (3,6)? The slope is still 2, and the equation would be this:

$$\begin{aligned} y &= mx + b \\ 6 &= 2 \cdot 3 + b \\ 6 &= 6 + b \\ 6 - 6 &= 6 + b - 6 \\ 0 &= b \end{aligned}$$

With either point on the line, the equation is the same: $y = 2x + 0$ or $y = 2x$.

Follow the solution steps in the next example.

Find the equation of the line that contains the points (–1,4) and (3,–8).

Solution: Find the slope of the line and then choose one of the given points for the *x* and *y* values in $y = mx + b$:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{4 - (-8)}{(-1) - 3} = \frac{4 + 8}{-4} =$$

$$\frac{12}{-4} = -3 \quad \rightarrow m = -3$$

Choose the point (–1,4). The equation becomes:

$$\begin{aligned} y &= mx + b \\ 4 &= -3 \cdot -1 + b \\ 4 &= 3 + b \\ 4 - 3 &= 3 + b - 3 \\ 1 &= b \end{aligned}$$

With $m = -3$ and $b = 1$, the equation for the line containing the points (–1,4) and (3,–8) is $y = -3x + 1$.

FYI Graphing the ordered pairs (–1,4) and (3,–8) confirms that the line going through these two points does have a *y*-intercept of +1.

A good way to verify this answer is to put the point you didn't choose, (3,–8), into the equation:

$$y = -3x + 1 \rightarrow -8 = -3 \cdot 3 + 1 \rightarrow -8 = -9 + 1 \rightarrow -8 = -8$$

The point (3,–8) does make the equation true and validates your answer.

Now try to write the equation for each of the following lines, checking your answers on page 217.

6. Find the equation of the line that contains the points (–3,–1) and (9,3).
7. Find the equation of the line that goes through (–1,8) and (2,–4).

Next you'll finish this hour by seeing some real-life applications of equations that include both *x* and *y* variables.

Solving Direct Variation Problems

In one of the earlier examples, the equation for the line that contains the points (1,2) and (3,6) is $y = 2x + 0$ or $y = 2x$. The ordered pair (1,2) shows that when $x = 1$, $y = 2$; the other ordered pair (3,6) shows that when $x = 3$, $y = 6$.

Can you predict the *y* value for the ordered pair if $x = 5$? The equation $y = 2x$ certainly helps you see that if $x = 5$, then $y = 10$, because putting 5 in for *x* gives the following equation:

$$y = 2x \rightarrow y = 2 \cdot 5 \rightarrow y = 10$$

Think about the real-life situation of paying $2 per jar for strawberry jam at the farmer's market. You know without doing any heavy algebra that three jars will cost $6, four jars will be $8, and so on.

Buying jars of this jam is a perfect example of *direct variation*. The cost (*y*) rises constantly as you buy more jars (*x*), and it rises at a constant rate—each additional jar you buy, you pay $2 more. Another way to say this is that *y* varies directly as *x*.

STRICTLY DEFINED

A **direct variation** equation is based on the slope-intercept formula $y = mx + b$, and b is always 0. Every direct variation equation can be written as $y = mx$. The effect is that as x increases, y also increases, and as x decreases, so does y. This "direct change" relationship between x and y is the reason for calling this siutation *direct variation.*

The cost of the jam is based on the algebra equation $y = 2x$, where x is the number of jars and y is the total cost, and there is no price break for volume purchases here.

Buying jars of jam can be described as an example of "y varies directly as x" because as x increases, so does y; as x decreases, so does y.

Think of the algebra side of this story: the equation $y = 2x$ has a slope (m) of 2 and a y-intercept (b) of 0. The graph of this equation would go through (0,0). With a slope of 2, the line would rise two units for every one unit of run because slope is the ratio of rise to run.

The line would rise to the right because the slope is positive, and the line would never end. There is an end to the number of jars the farmer can sell—when they're sold out for the day—but the graphed line goes on infinitely, in both directions.

The graphed line also includes all the ordered pairs between the whole number values, such as (2.5,5) and (3.4,6.8). In reality, you can't buy 2.5 jars of jam, but if you could, you would have to pay $5. A more realistic graph of the cost of the jam would be a series of dots beginning with the ordered pair (0,0)—if you buy no jars, you pay no money—and then (1,2), (2,4), (3,6), and so on.

When you can say that y varies directly as x, the general form of the equation is $y = kx$. When k is 5, the graphed line will be steeper than when k is 2, and y will increase more rapidly when k is 5 than when k is 2.

JUST A MINUTE

Recall that when a line's equation is in slope-intercept form—$y = mx + b$—the value of m for a line with a steeper slope will have a larger absolute value than the value of m for a line with a less steep slope. A line that is close to being horizontal has a slope between 0 and 1.

In many areas of research, business, and finance, direct variation plays an important role. For example, the weekly payroll costs for a company

vary directly as the total number of full-time employees who are working that week. Last week, the payroll was $31,500 for 45 employees who are each paid on the same pay scale. To meet the expected increase in orders for their product next week, five people who had been laid off during a slowdown are called in to work. What will the payroll costs be?

To set up this situation as a direct variation equation $y = kx$, you need to know what the k value is. You know that when x is 45, y is 31,500. Put these values into the equation and solve for k:

$$y = kx$$
$$31{,}500 = k \cdot 45$$
$$31{,}500 \div 45 = k \cdot 45 \div 45$$
$$700 = k$$

The equation for this situation is $y = 700x$.

To find the payroll costs for five additional people, the x value will be 50:

$$y = 700 \cdot 50 \rightarrow y = 35{,}000$$

The payroll costs for the 50 employees next week will be $35,000. This direct variation example depends on each employee receiving equal pay, which is not typical.

A more common direct variation problem is the cookie recipe example. The amount of chocolate chips in a batch of cookies varies directly with the number of cookies. For a large party coming up, you are going to bake 150 cookies, but your recipe will make only 60 cookies and calls for 2 cups of chocolate chips. What amount of chips will you need for the party recipe?

This is a direct variation problem, but be careful of the order. The chips vary directly with the number of cookies, so use $y = kx$, where $y = 2$ and $x = 60$. Solve for k:

$$2 = k \cdot 60$$
$$2 \div 60 = k \cdot 60 \div 60$$
$$\frac{2}{60} = k$$
$$\frac{1}{30} = k$$

The equation you will now use is $y = \frac{1}{30}x$. Put 150 in for x:

$$y = \frac{1}{30} \cdot 150$$
$$y = \frac{150}{30}$$
$$y = 5$$

You'll need 5 cups of chips for the 150 cookies.

FYI Another way to say that one item varies directly with another is to say that one is directly proportional to another. The same equation is used: $y = kx$.

Now try the following direct variation problems, checking your answers on page 218.

8. The amount you pay for a handyman to shovel snow from your sidewalks varies directly with the time it takes him to finish the job. If you paid him $30 for 20 minutes of his time, what will the bill be after a bigger snowstorm that takes him 50 minutes to clear?
9. The number of items that can be produced by a factory line varies directly with the number of hours the line is running. If the line can assemble 18 items in 4 hours, how many can be produced in 14 hours?

In many real-life situations, as one thing increases, another decreases. The following examples give you algebraic equations that apply in this type of variation problem.

Solving Inverse Variation Problems

When you order a large pizza to share with friends, the more people who want a piece, the smaller each piece will be. This is a great example of *inverse variation*. As the number of people increases, the size of the slice decreases, which is not at all like direct variation.

STRICTLY DEFINED

An **inverse variation** equation is based on the equation $xy = k$ and is often written as $y = k/x$. As x increases, y decreases, and as x decreases, y increases.

Consider the following plans for setting up the chairs for an outdoor wedding. In each row, the space between the chairs gets smaller as the number of chairs in the row increases. If 30 chairs fill the row when they are 12 inches apart, how many chairs can be in the row when they are 8 inches apart?

The inverse equation is $xy = k$. In this example, x is the number of chairs (30) and y is the space (12) between the chairs. Put 30 in the equation for x, 12 for y, and solve for k:

$$xy = k$$
$$30 \cdot 12 = k$$
$$360 = k$$

With $k = 360$, the equation for the chairs/spaces problem is $xy = 360$. Put 8 in for y to find the number of chairs:

$$xy = 360$$
$$x \cdot 8 = 360$$
$$x \cdot 8 \div 8 = 360 \div 8$$
$$x = 45$$

There can be 45 chairs in each row if they are 8 inches apart.

Another example of inverse variation follows:

> The speed of a car varies inversely as the time it is traveling. The faster the car's speed, the less time the trip takes. If your trip home from your friend's house takes 2 hours and you are going 50 miles per hour, how long will it take you to make the same trip if you go 60 miles per hour?

The inverse variation equation is $xy = k$. In this example, x is the speed of your car (50) and y is the time (2). Put 50 in the equation for x and 2 for y, and solve for k:

$$xy = k$$
$$50 \cdot 2 = k$$
$$100 = k$$

With $k = 100$, the equation for the problem is $xy = 100$. Put 60 in for x to find the time for the trip if you go 60 miles per hour:

$$60y = 100$$
$$60y \div 60 = 100 \div 60$$
$$y = 1.7 \text{ (rounded up)}$$

If you go 60 miles per hour, it will take you about 1.7 hours, and $1.7 \cdot 60$ is 102 minutes, so the time will be about 18 minutes less than the two hours it takes when you go 50 miles per hour.

Although the graph of the direct variation equation is always a straight line, the inverse variation graph is a curve. So the two types of variation not only apply to different situations, but their equations and graphs are also different.

Now try to solve this inverse variation problem, checking your answer on page 218.

10. When you use a crowbar to move an object, the force you apply varies inversely as the length of the crowbar—the shorter the crowbar, the more force you will need to apply.

If you must use 2 Newtons of force on a crowbar that is 6 feet long, how much force will it take if the crowbar is 4 feet long?

Answers to Sample Problems

1. Slope of line E $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \dfrac{4 - (-2)}{3 - 0} = \dfrac{4 + 2}{3} = \dfrac{6}{3} = 2 \rightarrow m = 2$; the y-intercept is $-2 \rightarrow b = -2$; putting $m = 2$ and $b = -2$ into $y = mx + b \rightarrow$ equation of line E is $y = 2x - 2$
2. Slope of line F $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \dfrac{3 - 0}{0 - (-1)} = \dfrac{3}{0 + (+1)} = \dfrac{3}{1} = 3 \rightarrow m = 3$; the y-intercept is $3 \rightarrow b = 3$; putting $m = 3$ and $b = 3$ into $y = mx + b \rightarrow$ equation of line F is $y = 3x + 3$

3. Slope of line G = –5 → m = –5; y-intercept is 8 → b = 8; putting m = –5 and b = 8 into $y = mx + b$ → equation of line G is $y = -5x + 8$

4. The ordered pair (–3,2) tells you that x = –3 and y = 2. The slope is –8, so m = –8. To find the b value, put x = –3, y = 2, and m = –8 into $y = mx + b$:

$$2 = -8 \cdot -3 + b$$
$$2 = 24 + b$$
$$2 - 24 = 24 + b - 24$$
$$-22 = b$$

m = –8 and b = –22 → equation of line is $y = -8x - 22$

5. The ordered pair (5,–6) tells you that x = 5 and y = –6. The slope is –3, so m = –3. To find the b value, put x = 5, y = –6, and m = –3 into $y = mx + b$:

$$-6 = -3 \cdot 5 + b$$
$$-6 = -15 + b$$
$$-6 + 15 = -15 + b + 15$$
$$9 = b$$

m = –3 and b = 9 → equation of line is $y = -3x + 9$

6. Find the slope (m) and y-intercept (b): slope = $\frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{-1-3}{(-3)-9} = \frac{-4}{-12} = \frac{1}{3} \rightarrow m = \frac{1}{3}$; from the point (9,3), x = 9, y = 3 → putting x = 9, y = 3, and $m = \frac{1}{3}$ into $y = mx + b$:

$$3 = \frac{1}{3} \cdot 9 + b$$
$$3 = 3 + b$$
$$3 - 3 = 3 + b - 3$$
$$0 = b$$

$m = \frac{1}{3}$ and b = 0 → equation of the line is $y = \frac{1}{3}x + 0$ or $y = \frac{1}{3}x$

7. Find the slope (m) and y-intercept (b): slope = $\frac{\text{rise}}{\text{run}} = \frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{8-(-4)}{-1-2} = \frac{8+(+4)}{-3} = \frac{12}{-3} = -4$ → m = –4; from the point (–1,8), x = –1, y = 8 → putting x = –1, y = 8 and m = –4 into $y = mx + b$:

$8 = -4 \cdot -1 + b$
$8 = 4 + b$
$8 - 4 = 4 + b - 4$
$4 = b$

$m = -4$ and $b = 4 \rightarrow$ equation of the line is $y = -4x + 4$

8. Amount you pay (y) varies directly with worker's time (x) $\rightarrow y = kx$:

$30 = k \cdot 20$
$30 \div 20 = k \cdot 20 \div 20$
$1.5 = k$

The equation is $y = 1.5x$; put 50 in for x:
$y = 1.5 \cdot 50 = 75 \rightarrow$ you will pay \$75 if it takes him 50 minutes to clear the snow.

9. Number of items (y) varies directly with number of hours (x) $\rightarrow y = kx$:

$18 = k \cdot 4$
$18 \div 4 = k \cdot 4 \div 4$
$4.5 = k$

The equation is $y = 4.5x$; put 14 in for x:
$y = 4.5 \cdot 14 = 63 \rightarrow$ 63 items can be assembled in 14 hours.

10. The force (x) varies inversely as the crowbar's length (y). The inverse variation equation is $xy = k$. Put 2 in the equation for x and 6 for y, and solve for k:

$xy = k$
$2 \cdot 6 = k$
$12 = k$

With $k = 12$, the equation for the problem is $xy = 12$. Put 4 in for y to find how much force you must apply:

$x \cdot 4 = 12$
$x \cdot 4 \div 4 = 12 \div 4$
$x = 3$

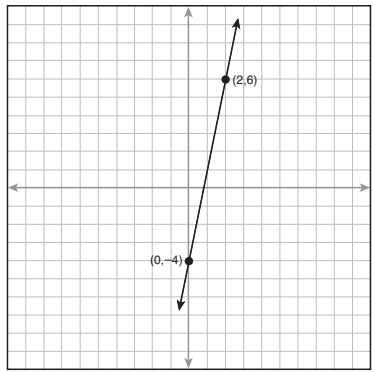
3 Newtons of force are needed on a 4-foot crowbar.

Hour's Up!

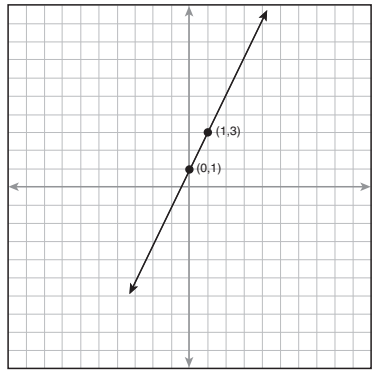
Based on what you've learned in Hour 16, do the following problems. Check your answers with the solution key in Appendix B.

Review

1.



2.



1. The equation of the line in graph 1 is

a. $y = 4x - 5$

b. $y = 5x - 4$

c. $y = 5x + 4$

d. $y = 4x + 5$

Review

2. The equation of the line in graph 2 is
a. $y = 3x + 2$
b. $y = 3x + 1$
c. $y = 2x + 1$
d. $y = 2x + 2$

3. The equation of the line with a slope of –6 and y-intercept of 9 is
a. $y = 9x - 6$
b. $y = 6x + 9$
c. $y = -6x - 9$
d. $y = -6x + 9$

4. The equation of the line with y-intercept of –5 and a slope of 3 is
a. $y = -5x + 3$
b. $y = 3x + 5$
c. $y = 3x - 5$
d. $y = 5x + 3$

5. The equation of the line that goes through the point (3,5) and has a slope of –2 is
a. $y = -2x + 11$
b. $y = 2x + 13$
c. $y = 2x - 11$
d. $y = -2x + 13$

6. The equation of the line that goes through the point (–2,8) and has a slope of 6 is
a. $y = 6x + 20$
b. $y = 6x - 50$
c. $y = -6x + 20$
d. $y = -6x - 50$

7. The equation of the line that contains the points (2,4) and (–1,–5) is
a. $y = -3x - 2$
b. $y = 3x - 2$
c. $y = 3x - 10$
d. $y = -3x - 10$

8. The gross amount of your weekly paycheck varies directly with the total dollar amount of sales you make in a gift shop. If your gross paycheck was $600 based on sales of $4,000, what should your paycheck be if you have $5,000 in sales for the week?
a. $650
b. $675
c. $700
d. $750

9. Your new car has been going about 360 miles on each 12-gallon fill up of gas, and the number of miles varies directly with the number of gallons of gas. How many miles can you expect to go on 10 gallons of gas?
a. 300
b. 320
c. 340
d. 280

10. The time it takes you to drive to your friend's house varies inversely as your average driving speed. (It will take more time to get there if your speed is slower.) The trip takes 3 hours if you drive an average speed of 45 miles per hour. How long will it take if you drive an average of 36 miles per hour due to rainy weather?
a. 3.50 hours, or 3 hours, 30 minutes
b. 3.75 hours, or 3 hours, 45 minutes
c. 4.25 hours, or 4 hours, 15 minutes
d. 4.50 hours, or 4 hours, 30 minutes

Hour 17

Solving Systems of Equations: Three Methods

CHAPTER SUMMARY

In this hour you will learn about …

- Solving a system of equations by graphing the lines
- Solving a system of equations by substitution
- Solving a system of equations by addition

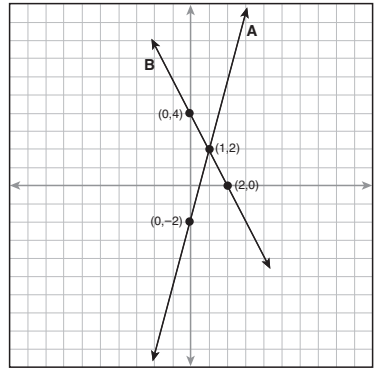
The world of two-variable equations that you entered in Hour 15 opens up possibilities of solving problems about direct and inverse variation, as well as being an efficient way to solve many problems that until now were limited to a single variable. This hour you'll see how to use your graphing skills to find any solutions that two equations may have in common. You'll also learn two algebraic methods that are very efficient alternatives to graphing. Let's begin with the graphing method.

Using Graphs to Solve a System of Equations

The equation $y = 4x - 2$ is graphed in the following figure and labeled line A. A second equation, $y = -2x + 4$, is also graphed on the same set of axes and labeled line B.

All of the points on each line represent the solutions for each of the equations. Notice the point (1,2), where the two lines cross. Because it is on both lines, the ordered pair (1,2) is a solution for both of the equations. Check this by putting $x = 1$ and $y = 2$ into each equation:

Line A: $y = 4x - 2 \rightarrow 2 = 4 \cdot 1 - 2 \rightarrow 2 = 4 - 2 \rightarrow 2 = 2$
Line B: $y = -2x + 4 \rightarrow 2 = -2 \cdot 1 + 4 \rightarrow 2 = -2 + 4 \rightarrow 2 = 2$



This confirms that (1,2) is a solution of both equations, and this will be helpful when you try to solve real-life problems that have two or more equations to describe a situation. The *system of equations* in this example is made up of $y = 4x - 2$ and $y = -2x + 4$, and you solve the system by finding the point where the two lines intersect: (1,2). This system has one solution because the lines meet at exactly one point.

STRICTLY DEFINED

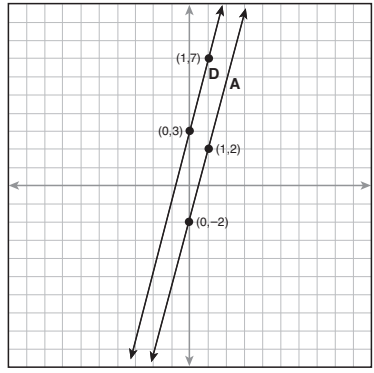
A **system of equations** is two (or more) equations that have two (or more) variables. The systems in this book have two equations in two variables. Three or more equations and variables are higher-level algebra topics.

Now consider the following system of equations and the graph of each line.

line A: $y = 4x - 2$ and line C: $y = 4x + 3$

The same line A is graphed on the following page, along with line C. Notice that there is no point where the two lines cross. When two lines have the same slope and different *y*-intercepts, they will not meet and are called parallel lines. If the lines don't cross, no point is on both the lines, so this system has no solution. No single ordered pair will work in both of these equations.

So far, you have seen an example of a system of equations that has one solution and a system with no solution. This third system might surprise you.

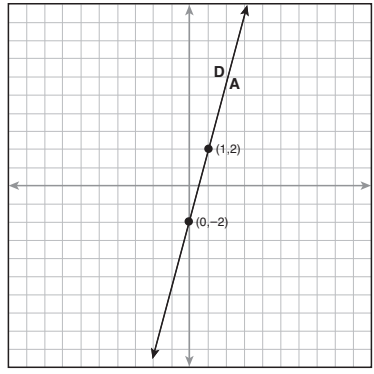


This system of equations is shown in the following graph:

line A: $y = 4x - 2$ and line D: $12x - 3y = 6$

JUST A MINUTE

To write an equation with *x* and *y* variables in slope-intercept form, you need to isolate the *y* variable or solve the equation for *y.* The properties of addition, subtraction, multiplication, and division help you move the terms so that you can see the *m* and *b* values and graph the line.



Again, line A is graphed here. On the graph, line D appears to be the same line. How can it be exactly the same line? Follow the steps of moving the terms in the equation for line D so that it is in slope-intercept form, or $y = mx + b$ form, which is the easiest way to graph an equation's line:

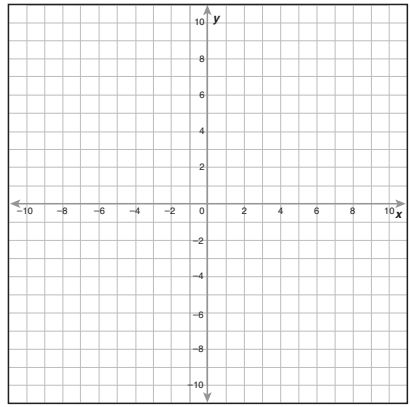
line D: $12x - 3y = 6$
$12x - 3y = 6$
$12x - 3y - 12x = 6 - 12x$
$-3y = 6 - 12x$
$-3y \div -3 = (6 - 12x) \div -3$
$y = 6 \div -3 - 12x \div -3$
$y = -2 + 4x$
$y = 4x + (-2)$
$y = 4x - 2$

As it turns out, the equation for line D, $12x - 3y = 6$, is the same as line A: $y = 4x - 2$. When the equations for lines A and D are graphed, the result is one line. All the points on the one line are solutions for each equation, so this system of equations has an infinite number of solutions.

FYI If solving an equation for y and writing it in slope-intercept form gives an equation that is identical to another equation, their graphs are the same line. Knowing that this system has an infinite number of solutions doesn't tell you about the solutions themselves. Graphing the line will show you many of the ordered pairs that are solutions to the system.

Now try solving the following system of equations by graphing. Check your answers on page 236.

1. Draw lines E and F on the following graph. Line E: $y = 2x - 4$ has a y-intercept of –4, has a slope of 2, and includes points (1,–2) and (2,0). Line F: $y = \frac{-1}{3}x + 3$ has a y-intercept of 3, has a slope of –1/3, and includes the points (3,2) and (6,1). Find the solution of this system of equations, if there is a solution.



Solving a system of equations by graphing the lines and finding where the lines cross—if they do cross—makes good use of your graphing skills. It gives you a visual representation of the system and its solution, whether there is one, many, or no solution.

However, there are some disadvantages to the graphing method. You might not have graph paper for drawing the lines. The equations might not have convenient slopes or *y*-intercepts that fit on the graph paper. The point where the two lines meet might not be whole numbers or might exist somewhere beyond the limits of your graph paper and be off the page so you can't see it.

For all these reasons, there are algebraic methods you can always count on to help you solve a system of equations. You'll learn about two of these methods in the next sections.

Solving a System of Equations by Substitution

When you are solving a system of equations, your goal is to find the solution, the ordered pair that makes both equations true. Graphing the two lines might show you the answer, but it is not very efficient, and you know by now that algebra is all about efficiency.

One of the basic principles of algebra is used so often that you might not realize it has a formal name. Every time you solve an equation and put the solution back into the original equation to verify that it is correct, you are using the *substitution property.*

STRICTLY DEFINED

The **substitution property** states that equal algebraic expressions can be substituted into (or used in place of) other algebraic expressions. For example, if $a = 4$, then $a + 7 = 4 + 7$; if $x = b + c$, then $6x = 6(b + c)$.

Consider the following system of equations:

$$y = 3x + 2 \text{ and } 2x + y = 12$$

You could graph the lines that represent these two equations to see where the lines meet, if they do meet. An alternative is to realize that the x and y variables appear in both equations, and the x in one is the same as the x in the other; the same is true of the y variables.

The substitution property lets you create a new equation based on the two equations in this system. The solution steps follow.

The two equations are $y = 3x + 2$ and $2x + y = 12$. In the first equation, y equals $3x + 2$. Therefore, $3x + 2$ can be substituted for y in the second equation, like this:

$2x + (3x + 2) = 12$ The ( ) just emphasize the substitution.

Now you can solve this new equation to find x:

$$\begin{aligned}
&2x + (3x + 2) = 12\\
&5x + 2 = 12\\
&5x + 2 - 2 = 12 - 2\\
&5x = 10\\
&5x \div 5 = 10 \div 5\\
&x = 2
\end{aligned}$$

Putting 2 in for x in either equation lets you find y. (The first equation is a little easier to use since the y term is already isolated, so use the first equation.)

$$y = 3x + 2 \rightarrow y = 3 \cdot 2 + 2 \rightarrow y = 6 + 2 \rightarrow y = 8$$

Now that you found $x = 2$ and $y = 8$, you know the ordered pair where the lines would intersect if you graphed them, and you know the solution to the system: (2,8)

To check your answer, put 2 in for x and 8 in for y in the second equation:

$$2x + y = 12 \rightarrow 2 \cdot 2 + 8 = 12 \rightarrow 4 + 8 = 12 \rightarrow 12 = 12$$

This confirms that (2,8) is the solution to the system of equations $y = 3x + 2$ and $2x + y = 12$.

Follow the substitution process as it efficiently finds the solution of this next system of equations.

FYI The equation with an isolated variable will be substituted into the other equation—it might be the first or second equation, and you might need to isolate a variable. For example, in the system $6x + y = 18$ and $y - 3x = 9$, the second equation can become $y = 3x + 9$ if you add $3x$ to both sides, and $3x + 9$ could be substituted for y in the first equation.

If $5x + y = -3$ and $y = x - 9$, solve this system.

In the second equation, y equals $x - 9$. Substitute $x - 9$ in the place of y in the first equation:

$$\begin{aligned}
&5x + (x - 9) = -3\\
&6x - 9 = -3\\
&6x - 9 + 9 = -3 + 9\\
&6x = 6\\
&6x \div 6 = 6 \div 6\\
&x = 1
\end{aligned}$$

Put 1 in for x in either equation (the second one is in an easier form, so use it) and solve for y:

$$y = x - 9 \rightarrow y = 1 - 9 \rightarrow y = -8$$

$x = 1$ and $y = -8$, so (1,–8) is the solution to the system of equations $5x + y = -3$ and $y = x - 9$.

When both equations have either isolated y variables or isolated x variables, the substitution method works especially well, as you see in the next example.

If $x = -3y + 6$ and $x = 2y - 9$, solve this system.

From the first equation, you know that x equals $-3y + 6$, so you can put $-3y + 6$ in the second equation where the x is, creating this new equation that can be solved for y:

$$\begin{aligned}
-3y + 6 &= 2y - 9 \\
-3y + 6 + 3y &= 2y - 9 + 3y \\
6 &= 5y - 9 \\
6 + 9 &= 5y - 9 + 9 \\
15 &= 5y \\
15 \div 5 &= 5y \div 5 \\
3 &= y
\end{aligned}$$

Put 3 in for y in the second equation and solve for x:

$$x = 2 \cdot 3 - 9 \rightarrow x = 6 - 9 \rightarrow x = -3$$

$x = -3$ and $y = 3$, so $(-3,3)$ is the solution to the system of equations $x = -3y + 6$ and $x = 2y - 9$.

The final example that follows shows a slightly more complex situation, but the process is still exactly the same.

If $y = 4x - 2$ and $6x - y = 8$, solve this system.

In the first equation, y equals $4x - 2$. Substitute $4x - 2$ in the place of y in the second equation:

$$6x - (4x - 2) = 8 \qquad \text{The ( ) emphasize the subtraction!}$$

Watch the signs as you subtract the binomial $4x - 2$. You must change the signs of both the $4x$ and the -2 when the subtraction changes to addition: subtracting $4x - 2$ is really adding $-4x + 2$.

GO TO ◄

Subtraction can be changed to addition if you change the signs of all the terms in the second expression (see Hour 3).

$$\begin{aligned}
6x - (4x - 2) &= 8 \\
6x + (-4x) + 2 &= 8 \\
2x + 2 &= 8 \\
2x + 2 - 2 &= 8 - 2 \\
2x &= 6 \\
2x \div 2 &= 6 \div 2 \\
x &= 3
\end{aligned}$$

Put 3 in for x in the first equation to solve for y:

$$y = 4x - 2 \rightarrow y = 4 \cdot 3 - 2 \rightarrow y = 12 - 2 \rightarrow y = 10$$

$x = 3$ and $y = 10$, so (3,10) is the solution to the system of equations $y = 4x - 2$ and $6x - y = 8$.

Now try to solve these systems of equations using the substitution method. Check your answers on page 236.

2. If $y = 3x - 4$ and $x + y = 16$, solve this system.
3. If $4x - y = 3$ and $y = -5x - 12$, solve this system.

When you are solving a system of equations that includes an equation with one of the variables isolated, as in the previous examples, substitution works very well. If the two equations have no isolated variables, there is still a great alternative to graphing the lines. Let's finish this hour with another algebraic method.

Solving a System of Equations by Addition

Back in Hour 5, you learned that the addition property is helpful when you want to solve an equation like $x - 5 = 3$. Without much hesitation, you would add 5 to both sides and solve for x:

$$x - 5 + 5 = 3 + 5$$
$$x = 8$$

The addition property of equality states that adding the same number to equal expressions gives equal sums. This property isn't limited to helping you move numbers from one side of an equation to the other to isolate a variable, as in the previous simple equation.

Remember that the equals sign is the powerful symbol that connects two expressions that are equal to each other. The addition property says that when you add equal items to equal expressions, you have expressions that remain equal.

JUST A MINUTE

Terms can be combined with adding and subtracting when they are like terms. They must be the same variable with equal exponents (in these equations, all the variables have exponents of 1). Their coefficients (the number in front of each variable) are then combined. Exponents never change when terms are added or subtracted.

For example, if $x + y = 8$ and $5x + 2y = 6$, the addition property allows these equations to be added together. The expressions on the left side of the equals sign are combined to produce $6x + 3y$, and the expressions on the right side of the equals sign are combined to produce a sum of 14. Therefore, $6x + 3y = 14$ is the result when adding these two equations together. The like terms are combined, and no terms are moved or rearranged in any way. In this particular example, it isn't really of great help that the two separate equations created the new one—$6x + 3y = 14$ is still an equation with two variables, just as the two equations were to begin with.

However, adding certain equations together will result in a very convenient new equation, as in the next example:

Consider the system of equations $x + y = 30$ and $x - y = -6$. You can graph the two lines to see where they cross, but the first equation has a y-intercept of 30, so the line won't fit on a typical coordinate graph numbered from –8 to +8. Substitution would also work, but you would have to isolate one of the variables first. There is a better way, and it's all about addition.

You need to add these two equations, and the clearest way to see the adding process is to stack the equations vertically, as they are shown here:

$$\begin{array}{r} x + y = 30 \\ + \underline{x - y = -6} \\ 2x + 0 = 24 \end{array}$$

The y variables sum to 0, and the new equation has only the x variable. Solving for x is easily done: divide both sides by 2, and the rest of the solution follows the same process as in the substitution method.

$$\begin{aligned} 2x &= 24 \\ 2x \div 2 &= 24 \div 2 \\ x &= 12 \end{aligned}$$

Put 12 in for x in the first equation and solve for y:

$$12 + y = 30 \rightarrow 12 + y - 12 = 30 - 12 \rightarrow y = 18$$

$x = 12$ and y = 18, so (12,18) is the solution to the system of equations $x + y = 30$ and $x - y = -6$.

TIME SAVER

You have learned how to use three methods for solving systems of equations: graphing, substitution, and addition. Decide which is most efficient based on the type of equations in each system.

You have three options when you are solving systems of equations, and addition is the perfect choice when the coefficients of either the x or y variables are opposites.

Adding opposites always results in 0, so this is an effective way to eliminate one of the variables in a system of equations. The following example shows detailed solution steps in a system of equations with opposite coefficients on the x variables:

If $-3x - 2y = -19$ and $3x - 5y = 5$, solve this system.

Solution: $-3x$ and $3x$ are opposites. Stack the equations vertically and add them to make a new equation with only the y variable:

$$\begin{array}{l} -3x-2y=-19 \\ +\ \underline{3x-5y=\ 5} \\ \ \ -7y=-14 \end{array}$$

Dividing both sides of this equation by $-7 \rightarrow y = 2$.

Put 2 in for y in the second equation:

$$3x - 5y = 5 \rightarrow 3x - 5 \cdot 2 = 5 \rightarrow 3x - 10 = 5 \rightarrow 3x - 10 + 10 = 5 + 10 \rightarrow 3x = 15 \rightarrow 3x \div 3 = 15 \div 3 \rightarrow x = 5$$

$x = 5$ and $y = 2$, so (5,2) is the solution to the system of equations $-3x - 2y = -19$ and $3x - 5y = 5$.

Now try to solve the following systems of equations with the addition method. Check your answers on page 236.

4. If $x - y = 9$ and $x + y = 5$, solve this system of equations.
5. If $3x + 13y = 9$ and $-3x + 5y = 45$, solve this system of equations.

FYI Parallel lines have equal slopes (m) and different y-intercepts (b). If you write two equations in a system in the slope-intercept form of $y = mx + b$ and their m values are equal and their b values are not—then you know there is no solution to the system.

At the beginning of this hour, the graphing method of solving systems included a system of equations that had no solution because the lines did not cross but were parallel. Also, two of the equations made graphs of exactly the same line, so there were infinitely many solutions—all the points on that line. It is possible to have no solution or many solutions when you use the substitution or addition methods, as you will see in the next two examples.

If $y = 3x - 2$ and $3x - y = 8$, solve the system.

Solution: Because the y variable is isolated in the first equation, use substitution. Put $3x - 2$ in for y in the second equation and solve for x:

$$3x - (3x - 2) = 8$$
$$3x + (-3x) + 2 = 8$$
$$0 + 2 = 8$$
$$2 = 8$$

However, 2 cannot equal 8, so there is no solution to this system of equations—no ordered pair exists that makes both of these equations true at the same time. If you do graph these lines, they will be parallel and will confirm that there is no solution.

JUST A MINUTE

Parallel lines have equal slopes and different y-intercepts, and they never cross; no ordered pair exists that can make both the equations true if the equations graph into parallel lines.

If $2x + y = 4$ and $y = -2x + 4$, solve the system.

Solution: The y variable is isolated in the second equation, so use substitution. Put $-2x + 4$ in for y in the first equation and solve for x:

$$2x + (-2x + 4) = 4. \quad \text{The ( ) show } -2x + 4 \text{ being put in for } y.$$
$$0 + 4 = 4$$
$$4 = 4$$

4 does always equal 4, so this system has infinitely many solutions. If you graph these two lines, they will form exactly the same line, and the solutions to the system are all the points on this line. You might also note that if you solve the first equation for y by subtracting $2x$ from both sides, the equation becomes exactly the same as the second equation. This confirms that there are infinitely many solutions to this system of equations.

Now try to solve this system of equations. It will have either infinitely many solutions or no solution. Check your answer on page 237.

6. If $6x + y = 8$ and $y = -6x + 3$, solve this system of equations.

To summarize this hour of solving systems of equations:

Graphing can show the solution to a system, but it is only the preferred method if both of the equations are in slope/intercept form, you have access to graph paper, and the slope and y-intercept values are simple and can fit on your graph.

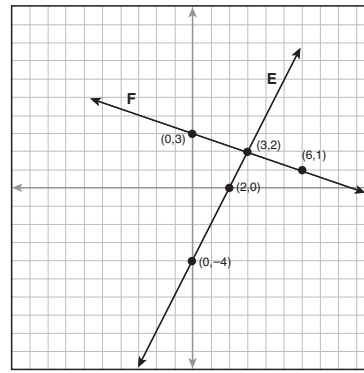
Substitution is the preferred method if one—or both—of the equations has an isolated variable, allowing you to substitute what it equals into the other equation in that variable's place.

Addition is the preferred method if the coefficients of the x or y variables are opposites. This allows you to add the two equations, and the opposites will sum to 0, creating a new equation in one variable that is easily solved.

In both substitution and addition, remember to put the first known variable's value back into one of the equations to solve for the second variable. Your goal is to find the ordered pair that makes both the equations in the system true at the same time, so you need to find both the x and y values that make the equations true.

Answers to Sample Problems

1. The lines are graphed in the following, and the solution of this system is (3,2).



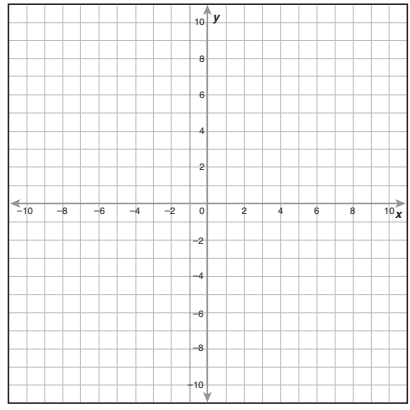
2. Put $3x - 4$ into the second equation for $y \rightarrow x + (3x - 4) = 16 \rightarrow 4x - 4 = 16 \rightarrow 4x - 4 + 4 = 16 + 4 \rightarrow 4x = 20 \rightarrow 4x \div 4 = 20 \div 4 \rightarrow x = 5$; put 5 in for x in the first equation and solve for y: $y = 3 \cdot 5 - 4 \rightarrow y = 15 - 4 \rightarrow y = 11$. Solution is (5,11).
3. Put $-5x - 12$ into the first equation for $y \rightarrow 4x - (-5x - 12) = 3 \rightarrow 4x + 5x + 12 = 3 \rightarrow. 9x + 12 = 3 \rightarrow 9x + 12 - 12 = 3 - 12 \rightarrow 9x = -9 \rightarrow 9x \div 9 = -9 \div 9 \rightarrow x = -1$; put -1 in for x in the first equation and solve for y: $4 \cdot -1 - y = 3 \rightarrow -4 - y = 3 \rightarrow -4 - y + 4 = 3 + 4 \rightarrow -y = 7 \rightarrow -y \div -1 = 7 \div -1 \rightarrow y = -7$. Solution is (–1,–7).
4. $x - y = 9$
   $+ \underline{x + y = 5}$
   $2x = 14$ Divide both sides by $2 \rightarrow x = 7$; put 7 in for x in the second equation. Solve for y: $7 + y = 5 \rightarrow 7 + y - 7 = 5 - 7 \rightarrow y = -2$. Solution is (7,–2).
5. $3x + 13y = 9$
   $+ \underline{-3x + 5y = 45}$
   $18y = 54$ Divide both sides by $18 \rightarrow y = 3$; put 3 in for y in the first equation and solve for x: $3x + 13 \cdot 3 = 9 \rightarrow 3x + 39 = 9 \rightarrow 3x + 39 - 39 = 9 - 39 \rightarrow 3x = -30 \rightarrow 3x \div 3 = -30 \div 3 \rightarrow x = -10$. Solution is (–10,3).

6. Use substitution: put $-6x + 3$ into the first equation for y and solve for x: $6x + (-6x + 3) = 8 \rightarrow 0 + 3 = 8 \rightarrow 3 = 8$. However, 3 cannot equal 8, so this system has no solution.

Hour's Up!

Based on what you've learned in Hour 17, find the solutions for the following systems of equations using the graphing, substitution, or addition methods. Check your answers with the solution key in Appendix B.

1. line G: $y = x - 3$ and line H: $y = -2x + 3$
The slope of line G is 1 and points include (0,–3) and (3,0); the slope of line H is –2 and points include (1,1) and (0,3).

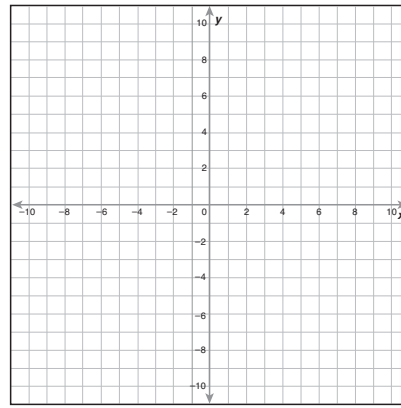


a. (1,–2)
b. (2,–1)
c. Infinitely many solutions
d. No solution

Review

Review

2. line I: $y = 2x + 3$ and line J: $y = 2x - 1$
The slope of line I is 2 and points include (0,3) and (-1,1); the slope of line J is 2 and points include (0,–1) and (2,3).



a. (1,0)
b. (0,3)
c. Infinitely many solutions
d. No solution

3. $y = -2x + 2$ and $3x + y = 5$
a. (–3,4)
b. (3,–4)
c. Infinitely many solutions
d. No solution

4. $x - 3y = 8$ and $x = 2y + 7$
a. (5,–1)
b. (2,3)
c. Infinitely many solutions
d. No solution

5. $y = 5x + 1$ and $5x - y = 3$
a. (–1,3)
b. (3,–1)
c. Infinitely many solutions
d. No solution

6. $4x + y = 6$ and $y = 6 - 4x$
a. (6,6)
b. (0,6)
c. Infinitely many solutions
d. No solution

7. $2x + 3y = 6$ and $4x - 3y = 12$
a. (3,0)
b. (0,3)
c. Infinitely many solutions
d. No solution

8. $3x - 2y = 1$ and $-3x + 2y = 7$
a. (0,8)
b. (8,0)
c. Infinitely many solutions
d. No solution

9. $2x - y = 5$ and $-x + y = -3$
a. (1,–2)
b. (2,–1)
c. Infinitely many solutions
d. No solution

10. $3x - 2y = -6$ and $-2x + 2y = 2$
a. (–4,–3)
b. (–3,–4)
c. Infinitely many solutions
d. No solution

Hour 18

Solving Systems of Equations: Combined Operations

CHAPTER SUMMARY

In this hour you will learn about …

- Solving a system of equations with a combination of multiplication and addition
- Solving practical problems with systems of equations

The three choices for solving a system of equations give you the flexibility to decide which is most appropriate for each system. If graphing equations as lines helps you see the solution in a clearer way than the algebraic options do, use your graphing skills. As you look at the two equations in a given system, the most efficient option will often be obvious. The substitution and addition methods let you use your algebra skills to solve these rather complex problems, but you can decide which of the three might work best in a given system.

Some systems of equations don't really fit well with graphing, substitution, or addition. The fourth and final option is a variation of the addition method, and it completes your package of skills for solving just about any system of equations.

Solving a System of Equations with the Multiplication/Addition Method

Being able to analyze a situation, consider possible actions, choose the best plan, and then carry out the plan correctly are all skills you develop as you study algebra. Each new layer of complexity builds on your knowledge, sometimes using very basic arithmetic skills alongside very sophisticated algebraic concepts. In this hour, you will see how fundamental arithmetic operations are helpful in solving complex algebra

problems, and at the end of the hour, you will be able to apply these new skills to solve real-life problems quite efficiently.

Consider the following system of equations:

$$2x - y = -8 \text{ and } 3x - y = -12$$

Graphing is not an efficient option here. Substitution could work, but neither of the equations has an isolated variable. Addition won't help, either, because none of the coefficients are opposites. Of the three choices, addition comes the closest to being the best choice—you just need to adjust one of the equations slightly.

Begin by stacking the equations vertically:

$$\begin{array}{r} 2x - y = -8 \\ +\ 3x - y = -12 \end{array}$$

JUST A MINUTE

The coefficients of either the *x* or *y* variables must be opposites for the variables to sum to 0 when added together. This allows you to solve the new equation for the other variable and find the solution for the system of equations.

Adding these two equations gives a new equation, but both the x and y variables are still there:

$$5x - 2y = -20$$

This new equation isn't helpful.

Look again at the stacked equations:

$$\begin{array}{r} 2x - y = -8 \\ +\ 3x - y = -12 \end{array}$$

It would be very helpful if one of the y variables could be changed from $-y$ to $+y$, because the y variables would be opposites and sum to 0, creating a new equation that could be solved for x. Multiplying $-y$ by -1 equals $+y$, which is the goal.

Although you can't simply multiply one term in an equation by –1, the multiplication property allows you to multiply a number times both

sides of any equation, and the equation will still be true. The following steps show the process of applying the multiplication property to the first equation in the system—although either equation could be used. The –1 multiplier is shown on the arrow after the first equation. The new first equation is $-2x + y = 8$. The second equation has no changes, so a multiplier of 1 is above its arrow, and the equation is simply copied.

JUST A MINUTE

Recall that your goal in solving a system by addition or substitution is eliminating one of the variables so that the new equation has only one variable in it. If the new equation still has two different variables, you need to multiply one of the equations by a number before adding the equations together.

$$\begin{array}{rcl} 2x - y = -8 & \xrightarrow{-1} & -2x + y = 8 \\ +\ 3x - y = -12 & \xrightarrow{1} & +\ 3x - y = -12 \\ & & x = -4 \end{array}$$

By multiplying all the terms in the first equation by –1, you get an equation that can be added to the second one, and the y variables sum to 0. In this problem, the new equation is already solved for x. (This doesn't always happen—you usually have to solve the new equation for x.)

To find the y value, put –4 into one of the original equations and solve it for y:

$$2 \cdot -4 - y = -8 \rightarrow -8 - y = -8 \rightarrow -8 - y + 8 = -8 + 8 \rightarrow y = 0$$

$x = -4$ and $y = 0$, so (–4,0) is the solution to the system of equations $2x - y = -8$ and $3x - y = -12$.

This process of solving a system of equations is a variation on the addition method you learned last hour. Multiplying an entire equation by a number that causes the x or y variables to sum to 0 when the equations are added is called the multiplication/addition method.

Let's look at another example of a system that can be solved with this new method:

$3x - 2y = -22$ and $x + 4y = 2$

Stacking the equations may help you see more clearly what needs to be done:

$$\begin{array}{r} 3x - 2y = -22 \\ +\ x + 4y = 2 \end{array}$$

Adding these equations as they are will not eliminate the x or y variables. What number can you multiply by one of the equations to make the x or y variables become opposites?

PROCEED WITH CAUTION

In many of the multiplication/addition systems, you will have more than one option of what multiplier makes the x or y variables sum to 0. List the choices and use the one that leads to fewer sign changes. You are less likely to make a careless error if you multiply by a positive number, but attention to detail is very important in all these problems.

You have two options: multiplying the second equation by –3 will make the x variables become opposites, and multiplying the first equation by 2 will make the y variables become opposites. It is your decision—either option will work.

When given the choice, however, you may want to choose multiplying by a positive number, just to avoid possible errors with the sign changes from multiplying by a negative number. The solution steps that follow show multiplication of the first equation by 2 and the second equation by 1, because it doesn't need to be changed:

$$\begin{array}{rl} 3x - 2y = -22 & \xrightarrow{2} \quad 6x - 4y = -44 \\ + \underline{x + 4y = 2} & \xrightarrow{1} \quad + \underline{x + 4y \quad = 2} \\ & \qquad\qquad 7x = -42 \end{array}$$

Dividing both sides by 7 solves the equation, and $x = -6$.

Put –6 in for x in the second equation—it is a slightly simpler equation than the first one, but either can be used—and solve for y:

$$\begin{aligned} -6 + 4y &= 2 \\ -6 + 4y + 6 &= 2 + 6 \\ 4y &= 8 \\ 4y \div 4 &= 8 \div 4 \\ y &= 2 \end{aligned}$$

$x = -6$ and $y = 2$, so (–6,2) is the solution to the system of equations $3x - 2y = -22$ and $x + 4y = 2$.

Consider the next system of equations and what you can do to solve it:

$3x + 4y = 12$ and $-x - 2y = 8$

Stacking them reveals the options:

$$\begin{aligned} &3x + 4y = 12 \\ +\ &{-x} - 2y = 8 \end{aligned}$$

You can multiply the second equation by 3 to make the x terms become opposites or by 2 to make the y terms become opposites. Either choice is about the same difficulty. Choosing the first option gives the following results:

$$\begin{aligned} &3x + 4y = 12 \xrightarrow{1} 3x + 4y = 12 \\ +\ &\underline{{-x} - 2y = 8} \xrightarrow{3} \underline{{-3x} - 6y = 24} \\ &\qquad\qquad\qquad\quad -2y = 36 \end{aligned}$$

Dividing both sides by –2 gives $y = -18$.

PROCEED WITH CAUTION

Always put the known value of the variable back into an original equation to solve for the other variable. Don't use one of the newer equations, since they may contain an error.

Putting in –18 for y in the first equation lets you avoid the negatives in the second equation:

$$\begin{aligned} &3x + 4 \cdot -18 = 12 \\ &3x - 72 = 12 \\ &3x - 72 + 72 = 12 + 72 \\ &3x = 84 \\ &3x \div 3 = 84 \div 3 \\ &x = 28 \end{aligned}$$

$x = 28$ and $y = -18$, so (28,–18) is the solution to the system of equations $3x + 4y = 12$ and $-x - 2y = 8$. This is a great example of a problem that would not be easy to graph because of the larger numbers, but the multiplication/addition method works very well.

Now try to solve the following systems with the multiplication/addition method. Check your answers on page 254.

1. $2x - y = -4$ and $-5x - y = 3$
2. $2x + y = 5$ and $3x - 2y = 4$

In all of the systems so far, you have been able to choose between two options in multiplying one of the equations by a number to make either the x or y variables sum to 0. The next system varies slightly from the others.

Solve the following system of equations:

$$5x + 2y = 14 \text{ and } 7x + 6y = 10$$

Begin by stacking them:

$$\begin{aligned} 5x + 2y &= 14 \\ +\ 7x + 6y &= 10 \end{aligned}$$

With all positive coefficients, you must multiply by a negative number to eliminate one of the variables. In all the earlier examples, both the x and y variables had common factors in their coefficients, giving you some choices. 5 and 7 have no common factors, but $2 \cdot 3$ equals 6, so multiplying 2 by –3 will eliminate the y variables in this system:

JUST A MINUTE

Finding the common factors of the coefficients helps you decide what multiplier will eliminate a variable. All numbers have a common factor of 1, but you are looking for common factors other than 1 to use as a multiplier.

$$\begin{aligned} 5x + 2y = 14 &\xrightarrow{-3} -15x - 6y = -42 \\ +\ \underline{7x + 6y = 10} &\xrightarrow{1} +\ \underline{7x + 6y = 10} \\ & \qquad -8x = -32 \end{aligned}$$

Dividing both sides by –8 gives $x = 4$.

Put 4 in for x in the first equation:

$$\begin{aligned} &5 \cdot 4 + 2y = 14 \\ &20 + 2y = 14 \\ &20 + 2y - 20 = 14 - 20 \\ &2y = -6 \\ &2y \div 2 = -6 \div 2 \\ &y = -3 \end{aligned}$$

$x = 4$ and $y = -3$, so (4,–3) is the solution to the system of equations $5x + 2y = 14$ and $7x + 6y = 10$.

Consider the following system of equations and its solution:

$$3x - 4y = -3 \text{ and } 2x - 3y = -1$$

Look for common factors in the coefficients:

$$\begin{array}{r} 3x - 4y = -3 \\ +\ 2x - 3y = -1 \end{array}$$

Similar to the last example, the x coefficients are both positive, and the y coefficients are both negative. Also, the coefficients of the x variable, 2 and 3, have no common factors, and –4 and –3 have no common factors.

The other examples have been simpler, in that one of the equations could be copied—multiplied by 1. This type of system needs multipliers other than 1 for each of the equations, to eliminate either the x or y variables.

Both equations will change, and you have two basic choices, but a negative multiplier is necessary in either choice. The choices for solving this system follow:

FYI In most multiplication/addition problems, you can choose from several options that will make the variables sum to zero. All the choices, if used correctly, will give the same ordered pair solution. It is a good idea to check your solution in both of the original equations in the system.

To eliminate the x variables, multiply the first equation by –2 and the second one by 3.

To eliminate the y variables, multiply the first equation by –3 and the second one by 4.

Both options are the same difficulty. If you choose the first and eliminate the x variables, the solution follows:

$$\begin{array}{rcl} 3x - 4y = -3 & \xrightarrow{-2} & -6x + 8y = 6 \\ +\ \underline{2x - 3y = -1} & \xrightarrow{3} & \underline{6x - 9y = -3} \\ & & -y = 3 \end{array}$$

Dividing both sides by –1 gives $y = -3$.

Put –3 in for y in the first equation and solve for x:

$$3x - 4 \cdot -3 = -3$$
$$3x + 12 = -3$$
$$3x + 12 - 12 = -3 - 12$$
$$3x = -15$$
$$3x \div 3 = -15 \div 3$$
$$x = -5$$

$x = -5$ and $y = -3$, so (–5,–3) is the solution to the system of equations $3x - 4y = -3$ and $2x - 3y = -1$.

Now try to solve the following systems of equations, checking your answers on pages 254–255.

3. $x + y = 5$ and $3x + 2y = 12$
4. $2x + 3y = -1$ and $3x + 5y = -2$

You can choose among the four methods for solving a system of equations. The algebra skills in this process combine with your ability to analyze a given problem and decide on a plan of action. The systems you have learned how to solve can also be based on real-life problems, as you will see in the final part of this hour.

JUST A MINUTE

Systems of equations can be solved by four methods: graphing, substitution, addition, and multiplication/addition. Choose the best one by analyzing the equations in the system.

Solving Practical Problems with Systems of Equations

The mixture problems that you set up with barrels and solved with a single variable in Hour 14 can also be written as a system of two equations. Look again at the produce department's granola mixture problem:

A store's produce department sells granola in bulk and makes its own custom mixture from a combination of granola and nut clusters. The granola costs $2 per pound, and the nut clusters cost $5 per pound. They want to make 60 pounds of the custom mixture that will cost $3 per pound. How much of the granola and of the nut clusters will it take?

The systems you have been solving all have x and y as their variables. When you are setting up a system for a real-world problem, x and y will always work, but using more meaningful letters helps you keep track of the solution more easily.

In this granola mixture problem, let g represent the pounds of granola and n represent the pounds of nut clusters. Setting up the system is just a matter of changing the sentences into equations, but a few helpful hints make the job easier:

FYI Mixture problems based on combining two items can be written as a system of two equations. One is usually the "counting" equation and involves summing the different variables. The other usually is the "value" equation and uses the individual price or weight of each item as the coefficient of each variable.

You will be writing two equations: one equation is about the amount being put into the mixture, and the other is about the value of the mixture. If you always do them in this order, the equation-writing process will be easier.

Here are the important facts in this problem:

The granola costs \$2 per pound, and the nut clusters cost \$5 per pound. They want to make 60 pounds of the custom mixture that will cost \$3 per pound. How much of the granola and of the nut clusters will it take?

The amount equation comes from this information:

> They want to make 60 pounds of the custom mixture that is made of granola plus nut clusters. $\rightarrow g + n = 60$

The first equation is $g + n = 60$.

The value equation comes from these sentences:

> The granola costs \$2 per pound, and the nut clusters cost \$5 per pound. The custom mixture will cost \$3 per pound.

Each pound of granola costs \$2, so g pounds will cost $2g$.

Each pound of nut clusters costs \$5, so n pounds costs $5n$.

All together, the new mixture will cost \$3, so 60 pounds costs $3 \cdot 60$, which is $180 \rightarrow 2g + 5n = 180$.

The second equation is $2g + 5n = 180$.

TIME SAVER

Use variables for real-life problems that come from the actual information. If the problem is about bananas and apples, use ***b*** and ***a*** instead of ***x*** and ***y*** to make the variables more meaningful to the situation.

Solve the system of equations: $g + n = 60$ and $2g + 5n = 180$

Graphing and substitution are not the easiest methods for this system, so stack the equations to analyze their coefficients:

$$\begin{array}{r} g + n = 60 \\ + 2g + 5n = 180 \end{array}$$

The multiplication/addition method will work perfectly. Your options are to multiply the first equation by –2 to eliminate the g variable or to multiply by –5 to eliminate the n variable. The following solution steps show the first option:

$$\begin{array}{rl} g + n = 60 & \xrightarrow{-2} \quad -2g + -2n = -120 \\ + \underline{2g + 5n = 180} & \xrightarrow{1} \quad \underline{2g + 5n = 180} \\ & \qquad\qquad 3n = 60 \end{array}$$

Dividing both sides by 3 gives $n = 20$.

Put 20 in for n in the first equation:

$$\begin{aligned} g + 20 &= 60 \\ g + 20 - 20 &= 60 - 20 \\ g &= 40 \end{aligned}$$

40 pounds of granola and 20 pounds of nut clusters will be in the new mixture.

Another typical algebraic problem about finding two unknown numbers is easily solved with a system of two equations, as in the following example.

GO TO ◄
Refer to the operations and symbols chart and their common English phrases in Hour 1.

The sum of two numbers is 80. One number is 8 more than 2 times the other. Find the numbers.

Let one number be x and the other number be y.

The first equation comes from the first sentence, "The sum of two numbers is 80." → $x + y = 80$

The second equation comes from the second sentence, "One number is 8 more than 2 times the other." $\rightarrow x = 8 + 2y$

Solve this system of equations:

$$x + y = 80 \text{ and } x = 8 + 2y$$

With the x variable isolated in the second equation, substitution is an easy method to use for this system. Put $8 + 2y$ into the first equation for x:

$(8 + 2y) + y = 80$ The parentheses emphasize the substitution.
$8 + 3y = 80$
$8 + 3y - 8 = 80 - 8$
$3y = 72$
$3y \div 3 = 72 \div 3$
$y = 24$

Put 24 in for y in the first equation and solve for x:

$x + 24 = 80$
$x + 24 - 24 = 80 - 24$
$x = 56$

The two numbers that meet the given conditions are 56 and 80; 56 and 24 sum to 80 and 56 is eight more than twice 24.

A slight variation of the number-search problem follows. One number is 18 more than another number. The sum of the smaller and twice the larger number is 45. Find the numbers.

In this problem, you need to specify which variable is the larger number and which is the smaller. It might help you to keep track of the numbers if you let x be the smaller and y be the larger—x is before y in the alphabet.

PROCEED WITH CAUTION

When one of the two numbers in the problem is larger than the other, set up the equation carefully. You must add to the smaller number to get the larger one. The common mistake is to add to the larger number because of the sentence's phrasing. Recall that "is" always means "equals," so put the equals sign in the place of "is" and write the rest of the equation around it.

The first equation comes from the first sentence, "One number is 18 more than another number." $\rightarrow y = 18 + x$

The second equation comes from the second sentence: "The sum of the smaller and twice the larger number is 45." $\rightarrow x + 2y = 45$

Solve this system of equations:

$y = 18 + x$ and $x + 2y = 45$

Substitution will work very well in this system, because the y is isolated in the first equation. Put $18 + x$ into the second equation for y:

$x + 2\ (18 + x) = 45$
$x + 2 \cdot 18 + 2 \cdot x = 45$
$x + 36 + 2x = 45$
$3x + 36 = 45$
$3x + 36 - 36 = 45 - 36$
$3x = 9$
$3x \div 3 = 9 \div 3$
$x = 3$

Put 3 in for x in the first equation and solve for y:

$y = 18 + 3 \rightarrow y = 21$

The numbers are 3 and 21, because 21 is 18 more than 3, and 3 plus two times 21 is 45.

Back to the real world—to calculate the air removed by ductwork in an HVAC system, use a system of equations.

Two air ducts together remove 600 cubic feet of air per minute. One duct removes 30 cubic feet of air per minute more than twice the air removed by the other duct. How much air per minute does each duct remove?

The complexity of this problem comes from the wording and your fear that you can't keep all the information straight. Take it one step at a time.

Decide what variables to use—x and y are fine. Begin with the first sentence, and write the first equation:

Two air ducts together remove 600 cubic feet of air per minute.
$\rightarrow x + y = 600$

The second equation comes from the second sentence: "One duct removes 30 cubic feet of air per minute more than twice the air removed by the other duct."

FYI Underlining or highlighting the key words in a story problem helps you focus on the math terms and removes the extra words that are not used to create the equation.

This sentence is the difficult one, but work through it a few words at a time. The italicized words are the key math words:

One duct $\rightarrow x$
removes $\rightarrow =$
30 cubic feet of air per minute *more than twice* the air removed by *the other duct* $\rightarrow$ 30 more than twice the other duct $\rightarrow 30 + 2y$

Put the sentence back together to write the equation:

$$x = 30 + 2y$$

The system of equations for this problem is:

$x + y = 600$ and $x = 30 + 2y$

Substitution is a good method because x is isolated in the second equation. Put $30 + 2y$ into the first equation for x:

$$(30 + 2y) + y = 30 + 3y = 600$$
$$30 + 3y - 30 = 600 - 30$$
$$3y = 570$$
$$3y \div 3 = 570 \div 3$$
$$y = 190$$

Put 190 in for y in the second equation and solve for x:

$$x = 30 + 2 \cdot 190 \rightarrow x = 30 + 380 \rightarrow x = 410$$

One of the ducts removes 410 cubic feet of air per minute, and the other removes 190 cubic feet of air per minute.

Now try to solve the following problems, checking your answers on page 255.

5. To complete a job estimate, you need to combine one material costing $2 per cubic foot with another material costing $6 per cubic foot to get 140 cubic feet of final product that will cost $3 per cubic foot. How much of each material should you use?
6. The current enrollment at a school is 860 students. There are 52 more girls than boys. How many boys are enrolled?

Answers to Sample Problems

1. $2x - y = -4 \xrightarrow{-1} -2x + y = 4$
   $+ -5x - y = 3 \xrightarrow{2} -5x - y = 3$
   $-7x = 7$
   Dividing both sides by –7 gives $x = -1$.
   Put –1 in for x in the first equation, and solve for y:
   $2 \cdot -1 - y = -4$
   $-2 - y = -4$
   $-2 - y + 2 = -4 + 2$
   $-y = -2$
   $-y \div -1 = -2 \div -1$
   $y = 2$
   (–1,2) is the solution to the system of equations.
2. $2x + y = 5 \xrightarrow{2} 4x + 2y = 10$
   $+ 3x - 2y = 4 \xrightarrow{1} 3x - 2y = 4$
   $7x = 14$
   Dividing both sides by 7 gives $x = 2$.
   Put 2 in for x in the first equation, and solve for y:
   $2 \cdot 2 + y = 5$
   $4 + y = 5$
   $4 + y - 4 = 5 - 4$
   $y = 1$
   (2,1) is the solution to the system of equations.
3. $x + y = 5 \xrightarrow{-2} -2x - 2y = -10$
   $+ 3x + 2y = 12 \xrightarrow{1} + 3x + 2y = 12$
   $x = 2$
   Put 2 in for x in the first equation, and solve for y:
   $2 + y = 5$
   $2 + y - 2 = 5 - 2$
   $y = 3$
   (2,3) is the solution to the system of equations.

4. $2x + 3y = -1 \xrightarrow{-3} -6x - 9y = 3$
$+ 3x + 5y = -2 \xrightarrow{2} + 6x + 10y = -4$
$y = -1$
Put –1 in for y in the first equation, and solve for x:
$2x + 3 \cdot -1 = -1$
$2x - 3 = -1$
$2x - 3 + 3 = -1 + 3$
$2x = 2$
$2x \div 2 = 2 \div 2$
$x = 1$
(1,–1) is the solution to the system of equations.
5. x is the material costing \$2 per cubic foot.
y is the material costing \$6 per cubic foot.
$x + y = 140 \xrightarrow{-2} -2x - 2y = -280$
$+ 2x + 6y = 3 \cdot 140 \xrightarrow{1} 2x + 6y = 420$
$4y = 140$
Dividing both sides by 4 gives $y = 35$. Put 35 into the first equation for y, and solve for x:
$x + 35 = 140$
$x + 35 - 35 = 140 - 35$
$x = 105$
You will need 105 cubic feet of the \$2 material and 35 cubic feet of the \$6 material.
6. b represents the number of boys, and g represents the number of girls
$b + g = 860$ and $g = b + 52$
Substitution method: put $b + 52$ into the first equation for g, and solve for b:
$b + (b + 52) = 860$
$2b + 52 = 860$
$2b + 52 - 52 = 860 - 52$
$2b = 808$
$2b \div 2 = 808 \div 2$
$b = 404$
There are 404 boys enrolled at the school.

Review

Hour's Up!

Based on what you've learned in Hour 18, find the solutions for the following systems of equations using the graphing, substitution, addition, or multiplication/addition method. Check your answers with the solution key in Appendix B.

1. $3x + y = 9$ and $-7x + y = -21$
   - **a.** (–3,0)
   - **b.** (0,3)
   - **c.** (3,0)
   - **d.** No solution
2. $2x + y = 3$ and $3x + y = 4$
   - **a.** (1,0)
   - **b.** (1,1)
   - **c.** (–1,–1)
   - **d.** No solution
3. $5x - 3y = 6$ and $-3x + y = -6$
   - **a.** (3,0)
   - **b.** (1,1)
   - **c.** (3,3)
   - **d.** No solution
4. $3x + 2y = 35$ and $4x - 5y = 16$
   - **a.** (3,6)
   - **b.** (9,4)
   - **c.** (4,9)
   - **d.** No solution
5. $3x - 2y = -1$ and $4x - 3y = -3$
   - **a.** (3,5)
   - **b.** (–7,7)
   - **c.** (3,–5)
   - **d.** No solution
6. $3x + 4y = -25$ and $2x - 3y = 6$
   - **a.** (–3,4)
   - **b.** (3,5)
   - **c.** (–3,–4)
   - **d.** No solution

7. A community theater production sold adult tickets for $6 each and child tickets for $4 each. They sold a total of 620 tickets for a total of $3,160. How many child tickets were sold?
a. 340
b. 280
c. 620
d. 380

8. To make 150 pounds of a nut mixture that will cost $5 per pound, how many pounds of cashews that cost $6 per pound and pecans that cost $3 per pound will it take?
a. 30 pounds of cashews and 120 pounds of pecans
b. 50 pounds of cashews and 100 pounds of pecans
c. 100 pounds of cashews and 50 pounds of pecans
d. 75 pounds of cashews and 75 pounds of pecans

9. The enrollment at a school is currently 620 students. There are 16 more boys than girls. How many girls are enrolled?
a. 318
b. 302
c. 298
d. 294

10. One number is 32 more than another number. The sum of the larger number and twice the smaller number is 59. What are the numbers?
a. 9 and 41
b. 8 and 40
c. 7 and 39
d. 6 and 38

Review

Hour 19

Solving Inequalities: Graphing

CHAPTER SUMMARY

In this hour you will learn about …

- Solving and graphing inequalities
- Solving and graphing combined inequalities

Learning how to solve systems of equations takes you into the two-dimensional world of ordered pairs and graphed lines with slopes and *y*-intercepts. The choices of how you find the solution for a system put you in charge, giving you the power to make a decision based on analyzing your options.

This hour, you return to solving problems with just one variable, but it's not really a step backward. It's taking the basic properties that help you solve equations and applying these same properties to solve problems you were introduced to in Hour 2—inequalities.

Solving and Graphing Inequalities

Many situations in life require an exact answer, including "How much acid should I add to 30 liters of 20% acidic solution to make it 40% acidic?" At other times, knowing a range of answers is more helpful, such as "What's the least amount of money I must deposit to be sure my checking account isn't overdrawn?"

In the acid solution example, setting up and solving an equation helps you find the answer. An equation always has an equals sign. If there is a variable in the equation, your job is to find the value of the variable that makes the equation true.

GO TO ◄
The acid solution problem is solved as an example in Hour 14.

In the checking account example, setting up and solving an inequality is the key. An inequality can have any of the following signs:

$<, \leq, >, \geq$

If there is a variable in the inequality, your job is to find the value(s) of the variable that make(s) the inequality true.

Fortunately, solving inequalities is similar to solving equations. Many of the same properties are used, with just a few variations that are not difficult to master.

Let's begin by reviewing the four inequality symbols, shown here in word and symbol examples:

Word Phrase	Inequality Expression
x is less than 5	$x < 5$
x is less than or equal to 5	$x \leq 5$
x is greater than 5	$x > 5$
x is greater than or equal to 5	$x \geq 5$

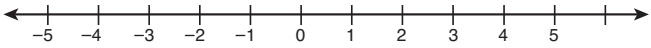
JUST A MINUTE

An easy way to remember < means "less than" is to notice that < looks like an *L* and "less than" begins with an *L*.

When you saw these symbols in Hour 2, they appeared in expressions like the following:

$2 > -3$ Is this a true statement? (yes)
$3 \geq 3$ Is this a true statement? (yes)
$-4 > 4$ Is this a true statement? (no)

These inequalities can be visualized and better understood when they are graphed on a number line:



JUST A MINUTE

The real number line extends from negative infinity on the left side to positive infinity on the right side. Zero separates the positive and negative numbers, and zero is the same distance from –1 as it is from +1.

$2 > -3$ is true because 2 is to the right of –3.

$3 \geq 3$ is true because "greater than OR equal to" allows the numbers to be equal.

$-4 > 4$ is false because –4 is to the left of 4; therefore,

$-4 < 4$ is the true statement.

Your introduction to equations in Hour 5 included three types of equations:

True → $2 + 3 = 5$
False → $2 + 3 = 6$
Open → $x + 3 = 5$ → Open equations can be true or false, depending on what number is put in for the variable.

Inequalities have similar types:

True → $2 + 3 \leq 5$
False → $2 + 3 < 5$
Open → $x + 3 < 5$ → As with equations, an open inequality might be true or false, depending on what number is put in for the variable.

Let's begin by considering more closely the example of the open inequality:

$$x + 3 < 5$$

If it were written as an open equation, it would be this:

$$x + 3 = 5$$

Solving it mentally is easy → $x = 2$.

The algebraic solution step involves the subtraction property:

JUST A MINUTE

The subtraction property lets you subtract the same amount from both sides of a true equation to create a new equation that is also true.

$x + 3 - 3 = 5 - 3$
$x = 2$

To check, put 2 in for x in the original equation:

$2 + 3 = 5 \rightarrow 5 = 5$; therefore, $x = 2$ is the solution.

Solving the inequality relies on the same subtraction property:

$x + 3 < 5$
$x + 3 - 3 < 5 - 3$
$x < 2$

To check, put a number that is less than 2 in for x in the original equation:

$1 + 3 < 5 \rightarrow 4 < 5$; therefore, $x < 2$ is the solution.

The subtraction property seems to apply to solving equations and inequalities. What if the inequality symbol is reversed? Try solving $x + 3 > 5$ with the subtraction property:

$x + 3 > 5$
$x + 3 - 3 > 5 - 3$
$x > 2$

To check, put a number greater than 2 in for x in the original equation:

$3 + 3 > 5 \rightarrow 6 > 5$; therefore, $x > 2$ is the solution.

Yes, the subtraction property does apply to solving equations and inequalities.

Consider another open inequality:

$x - 3 > 5$

If it were written as an open equation, it would be this:

$x - 3 = 5$

Solving it mentally is easy $\rightarrow x = 8$.

The algebraic solution step involves the *addition property*:

$$x - 3 = 5$$
$$x - 3 + 3 = 5 + 3$$
$$x = 8$$

JUST A MINUTE

The addition property lets you add the same amount to both sides of a true equation to create a new equation that is also true.

To check, put 8 in for x in the original equation:

$8 - 3 = 5 \rightarrow 5 = 5$; therefore, $x = 8$ is the solution.

Solving the inequality relies on the same addition property:

$$x - 3 > 5$$
$$x - 3 + 3 > 5 + 3$$
$$x > 8$$

To check, put a number that is greater than 8 in for x in the original equation:

$9 - 3 > 5 \rightarrow 6 > 5$; therefore, $x > 8$ is the solution.

The addition property applies to solving equations and inequalities.

A logical question at this point is how you know what number to put in for the variable when you check the inequality solution. The solution to the last inequality was $x > 8$, and 9 was chosen as the number. Would 10 have worked, or 11.8?

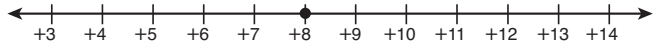
Yes, $10 - 3 > 5$ and $11.8 - 3 > 5$. When the solution is an inequality, such as $x > 8$, any number that is greater than 8 will be a solution. Looking at the graph for this solution may help you see this more clearly:

The solution is "all the numbers that are greater than 8." An open circle on the +8 shows that you are not including 8 in the solution. The arrow

to the right shows all those numbers—integers, fractions, decimals—that are to the right of 8. You can choose any of those numbers to put in the inequality for x, and each one will give a true inequality.

FYI In the same way that equations like $x + y = 6$ create a line with an infinite number of solutions on the (x,y) graph in Hour 15, inequality solutions create a line or ray on the real number line and show an infinite number of solutions.

Equation solutions can also be graphed on a number line. You know that the solution to the equation $x - 3 = 5$ is $x = 8$. Its graph is a single point—+8 on the number line:



A final example of an inequality that is solved with the addition property includes the ≤ symbol:

Solve: $x - 3 \leq 4$
Solution: Adding 3 to both sides isolates the x.
$x - 3 \leq 4$
$x - 3 + 3 \leq 4 + 3$
$x \leq 7$

To check, put a number less than or equal to 7 into the original inequality for x. 7 is less than OR equal to 7, so use 7:

$7 - 3 \leq 4 \rightarrow 4 \leq 4$ True: 4 is less than OR equal to 4.

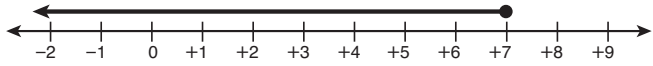
TIME SAVER

The number you choose to check your solution can be any number in the inequality: to check if $x > 8$ is the solution, choose 10 as the number to put in for x—it's greater than 8 and probably easier to work with than 9 is.

Now try 6, because 6 is less than or equal to 7:

$6 - 3 \leq 4 \rightarrow 3 \leq 4$ True: 3 is less than or equal to 4.

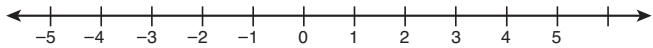
Graph this solution to see the answer more easily:



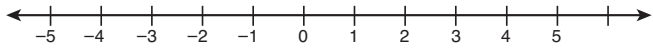
The graph is a line that begins solidly at +7 and goes to the left, because it includes all the numbers less than or equal to 7. The numbers on this solution line include all the integers, fractions, and decimals from 7 to negative infinity.

Now try to solve the following inequalities. Graph the solution on the number line, and check your answers on page 271.

1. $x + 5 < 7$



2. $x - 3 \geq -7$



The inequalities you solved so far have used the addition and subtraction properties. Before you work on solving inequalities that include multiplying or dividing, look at the following basic inequalities.

Consider the true inequality:

$6 < 12 \rightarrow$ It is true that 6 is less than 12.

What if you multiply both sides by 3?

$6 \cdot 3 < 12 \cdot 3 \rightarrow 18 < 36 \rightarrow$ True

What if you divide both sides by 3?

$6 \div 3 < 12 \div 3 \rightarrow 2 < 4 \rightarrow$ True

Multiplying or dividing both sides of this true inequality by positive 3 makes new inequalities that are still true.

JUST A MINUTE

The ordering of negative numbers can be confusing: recall that $-2 < -1$ and $-10 > -20$. Putting these numbers on a number line may help because numbers are always increasing as you move to the right along the number line and decreasing as you move to the left.

What if you multiply both sides by –3?

$$6 \cdot -3 < 12 \cdot -3 \rightarrow -18 < -36 \rightarrow \text{False!}$$

Dividing both sides by –3:

$$6 \div -3 < 12 \div -3 \rightarrow -2 < -4 \rightarrow \text{False!}$$

When you multiply or divide an inequality by a negative number, the inequality must be reversed to make it true.

Begin again with $6 < 12$.

You can multiply both sides by –3, but you must switch the inequality. "Less than" becomes "greater than," as follows:

$$6 < 12 \rightarrow \text{True}$$
$$6 \cdot -3 > 12 \cdot -3 \rightarrow -18 > -36 \rightarrow \text{True}$$

The < reversed to > to make a true inequality.

You can also divide both sides by –3, as long as you switch the inequality:

$$6 < 12 \rightarrow \text{True}$$
$$6 \div -3 > 12 \div -3 \rightarrow -2 > -4 \rightarrow \text{True}$$

The < reversed to > to make a true inequality.

This rule is very important in specific situations. If you are solving an inequality and either multiply or divide both sides to isolate a variable, you must watch for this reversal process.

You must switch the inequality symbol if you multiply or divide by a negative number.

The inequality symbol stays the same for all adding and subtracting, and for all multiplying or dividing by a positive number.

Let's consider an inequality that involves multiplication:

$$5x < 10$$

You can easily solve this with mental math: $x < 2$

The algebraic solution step uses the division property:

$$5x < 10$$
$$5x \div 5 < 10 \div 5$$
$$x < 2$$

JUST A MINUTE

The division property lets you divide both sides of a true equation by the same number to create a new equation that is also true. It also applies to inequalities, as long as the number you divide by is positive.

To check, put a number less than 2 in for x in the original inequality. Use 1:

$5 \cdot 1 < 10 \rightarrow 5 < 10$; therefore, $x < 2$ is the solution.

Notice that you divided by positive 5, so the < symbol stayed the same.

Follow the solution steps for the following inequality:

$$-3x < 18$$

Don't try solving this mentally, because the –3 needs to be divided on both sides, and the inequality must switch.

$$-3x < 18$$
$$-3x \div -3 > 18 \div -3 \quad \text{Dividing by } -3\text{: switch to } >$$
$$x > -6$$

To check, put a number greater than –6 into the original inequality for x. Use –5:

$-3 \cdot -5 < 18 \rightarrow 15 < 18 \rightarrow$ True

Therefore, $x > -6$ is the solution. It's important that you switch the inequality symbol only as you work to isolate the variable. The original inequality doesn't change. It can be difficult to keep track of these signs. Fortunately, it occurs only in the limited situation of multiplying or dividing by a negative number, but you must be aware of the importance of the effect.

The following inequality includes two operations, and the solution steps follow:

$2x - 8 \geq -14$
Solution: Your goal is to isolate the variable. Move the 8 by adding 8 to both sides, and then move the 2.
$2x - 8 + 8 \geq -14 + 8$
$2x \geq -6$
$2x \div 2 \geq -6 \div 2$
$x \geq -3$

(The inequality didn't switch—you didn't divide by a negative number.)

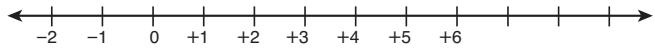
To check, put a number greater than or equal to –3 into the original inequality for x. Use 0:

$2 \cdot 0 - 8 \geq -14 \rightarrow 0 - 8 \geq -14 \rightarrow -8 \geq -14 \rightarrow$ True

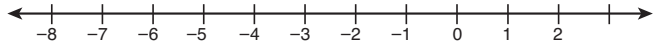
Therefore, $x \geq -3$ is the solution.

Now try to solve the following inequalities. Graph the solutions on the following number lines, and check your answers on pages 271–272.

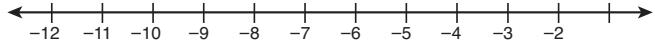
3. $6x - 3 < 21$



4. $-3x + 7 \leq 25$



5. $9 + 2x > -11$



The solutions to all the inequalities so far have been graphed lines that go to either positive or negative infinity from a specific beginning point. They give a visual picture of inequalities like $x < 5$, a very efficient way to say "all the numbers less than 5."

Sometimes you need to refer to a range of numbers. If you suspect you have a fever, you probably don't need to be concerned if your temperature is between 97.5° and 100.5°. You might need to see a doctor if it's above or below that range. The "safe range" can best be expressed with a new type of inequality, as you'll see in the next section.

Solving and Graphing Combined Inequalities

Describing a variable as having a value between two numbers can most efficiently be written as an interval, with inequality symbols setting the boundaries. The example of a non-emergency body temperature range is shown with the following statements and inequalities:

"It is acceptable for your temperature to be above 97.5" can be written like this:

$t > 97.5$; this can be reversed and be written $97.5 < t$.

PROCEED WITH CAUTION

The direction of an inequality reverses when the sides are reversed: $2 < 10 \rightarrow 10 > 2$; $x < 3 \rightarrow 3 > x$.

"It is acceptable for your temperature to be below 100.5" can be written like this:

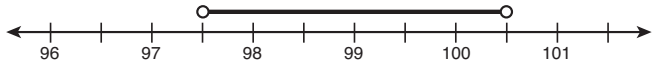
$t < 100.5$

When you combine these inequalities, the result is this:

$97.5 < t < 100.5$

In this combined form, the variable—your temperature—can be between 97.5° and 100.5°.

Graphing this interval makes it clear that t is between the two numbers on the graph, as well as in the combined inequality:



In the temperature example, the solution is the interval itself, because the t is already isolated.

Consider the following combined inequality:

$$8 \leq x - 4 \leq 15$$

To solve this interval, isolate the x by adding 4 to all three parts of the inequality:

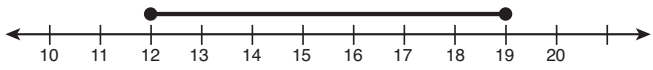
$$8 + 4 \leq x - 4 + 4 \leq 15 + 4$$
$$12 \leq x \leq 19$$

To check, choose a number between 12 and 19 to put into the original inequality for x. Use 15:

$$8 \leq 15 - 4 \leq 15 \rightarrow 8 \leq 11 \leq 15 \rightarrow \text{True}$$

Therefore, $12 \leq x \leq 19$ is the solution (all the numbers between 12 and 19, including 12 and 19).

The graph follows:



As with simpler inequalities, multiplying or dividing by a negative number reverses the inequality symbols:

$$-6 < -2x + 4 < 16$$

Solution: Isolate the variable by subtracting 4 from all three parts, and then divide all three results by –2.

$$-6 < -2x + 4 < 16$$
$$-6 - 4 < -2x + 4 - 4 < 16 - 4$$
$$-10 < -2x < 12$$
$$-10 \div -2 > -2x \div -2 > 12 \div -2$$
$$5 > x > -6$$

The preferred order for these interval solutions is to have the smaller number on the left side and the larger on the right—the same way the

number line is displayed. So each of the terms can be reversed as long as each inequality symbol is also reversed:

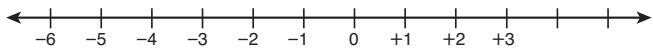
$$-6 < x < 5$$

The solution: "all numbers between –6 and 5"

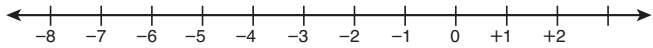
In this problem, the inequality symbols were reversed because of the division by –2, and then they were reversed again to put the smaller number on the left side of the expression. You have lots to keep track of here, but the process is always consistent, which is one of the certainties of algebra.

Now try to solve these combined inequalities, graphing the solutions and checking your answers on page 272.

6. $-4 \le 2x + 6 \le 10$

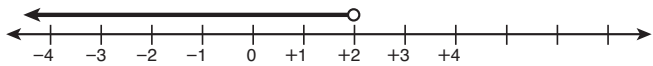


7. $-6 \le -3x - 9 \le 12$

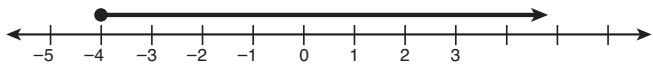


Answers to Sample Problems

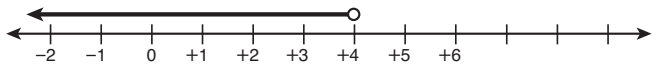
1. $x + 5 - 5 < 7 - 5 \rightarrow x < 2$



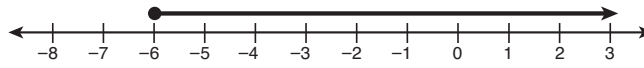
2. $x - 3 + 3 \ge -7 + 3 \rightarrow x \ge -4$



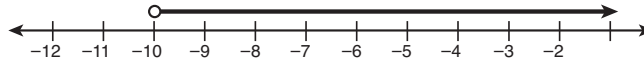
3. $6x - 3 + 3 < 21 + 3 \rightarrow 6x < 24 \rightarrow 6x \div 6 < 24 \div 6 \rightarrow x < 4$



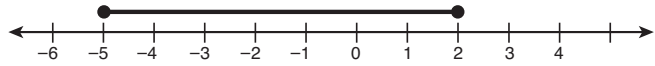
4. $-3x + 7 - 7 \leq 25 - 7 \rightarrow -3x \leq 18 \rightarrow -3x \div -3 \geq 18 \div -3 \rightarrow x \geq -6$



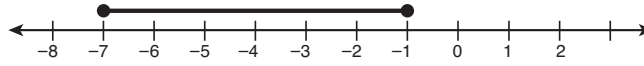
5. $9 + 2x - 9 > -11 - 9 \rightarrow 2x > -20 \rightarrow 2x \div 2 > -20 \div 2 \rightarrow x > -10$



6. $-4 \leq 2x + 6 \leq 10$
   $-4 - 6 \leq 2x + 6 - 6 \leq 10 - 6$
   $-10 \leq 2x \leq 4$
   $-10 \div 2 \leq 2x \div 2 \leq 4 \div 2$
   $-5 \leq x \leq 2$



7. $-6 \leq -3x - 9 \leq 12$
   $-6 + 9 \leq -3x - 9 + 9 \leq 12 + 9$
   $3 \leq -3x \leq 21$
   $3 \div -3 \geq -3x \div -3 \geq 21 \div -3$
   $-1 \geq x \geq -7$ (Reverse all terms and symbols.)
   $-7 \leq x \leq -1$



Hour's Up!

Review

Based on what you've learned in Hour 19, find the solutions for the following inequalities. Check your answers with the solution key in Appendix B.

1. $x - 5 > 3$
- **a.** $x > 8$
- **b.** $x < 8$
- **c.** $x > -8$
- **d.** $x < -8$

2. $x + 3 \leq 9$
- **a.** $x \geq -6$
- **b.** $x \geq 6$
- **c.** $x \leq -6$
- **d.** $x \leq 6$

3. $2x \geq -10$
- **a.** $x \leq 5$
- **b.** $x \leq -5$
- **c.** $x \geq -5$
- **d.** $x \geq 5$

4. $3x - 8 \leq 4$
- **a.** $x \leq 4$
- **b.** $x \leq -4$
- **c.** $x \geq 4$
- **d.** $x \geq -4$

5. $-5x > 10$
- **a.** $x < 2$
- **b.** $x < -2$
- **c.** $x > -2$
- **d.** $x > 2$

6. $-2x + 6 \leq 14$
- **a.** $x \geq -4$
- **b.** $x \geq 4$
- **c.** $x \leq -4$
- **d.** $x \leq 4$

Review

7. $-3x - 2 \leq 13$

a. $x \leq 5$

b. $x \leq -5$

c. $x \geq -5$

d. $x \geq 5$

8. $7 + 2x > 9$

a. $x < 1$

b. $x < -1$

c. $x > -1$

d. $x > 1$

9. $3 - 2x > 11$

a. $x < 4$

b. $x < -4$

c. $x > -4$

d. $x > 4$

10. $4x + 1 \leq -19$

a. $x \leq -5$

b. $x \leq 5$

c. $x \geq 5$

d. $x \geq -5$

Hour 20

Solving Inequalities: Absolute Value

CHAPTER SUMMARY

In this hour you will learn about …

- Solving and graphing absolute value equations
- Solving and graphing absolute value inequalities
- Graphing inequalities with two variables

The properties that helped you solve equations were also very useful in solving inequalities in the previous hour. Graphing the solutions on a number line made the infinite number of possible values for the variables easier to see. The number line is also convenient when you are working with absolute value, an algebraic way of finding how far a number is from 0.

This hour, you will see how to find the solution of equations and inequalities that include absolute value. The problems will be limited to single variable expressions, as they were in the last hour, and you'll see that most of the same properties you already know are used in these higher-level types of equations and inequalities.

Solving and Graphing Absolute Value Equations

The concept of absolute value is based on actual distance—how far a given point on the number line is from zero—but you'll also see equations that include the absolute value of a variable or other expression. Before you learn how to solve these equations, let's review the basics about absolute value.

Recall that the symbol for absolute value is two small vertical lines, | |, and that typically a number is inside:

$|-9| \rightarrow$ "the absolute value of negative 9"

JUST A MINUTE

When you learned about adding negative numbers in Hour 3, the absolute value of the number was used in the algebraic explanation. Example: (–7) + (–5) = –12 because the sum of two or more negative numbers is the negative sum of the numbers' absolute values, or $(-7) + (-5) \rightarrow -(|-7| + |-5|) \rightarrow -(7 + 5) \rightarrow -12$.

Because –9 is 9 units away from 0, $|-9| = 9$.

$|+9|$ is also 9 because +9 is 9 units away from 0.

The simplest way to know a number's absolute value is to simply remove its sign:

$$|-9| = |+9| = 9$$

An open equation that includes absolute value follows:

$$|x| = 6$$

As with all open equations, your job is to find the value of the variable that makes the open equation a true equation. What can x be in this open equation? Another way to ask the same question: What number is 6 units away from 0?

The two possibilities are that x can be –6 or +6 because if you put either –6 or +6 into the original equation, you get a true statement:

$$|-6| = 6 \text{ and } |+6| = 6$$

Also, both –6 and +6 are 6 units away from 0. No matter how you look at it, both numbers are the solution to $|x| = 6$, and this solution can be written as $x = 6$ or $x = -6$.

However, the best way to write this equation and its solution follows.

Solve: $|x| = 6$

Solution: $x = \pm 6$

This gives the solution in its most efficient form. You will hear it read aloud as "x equals positive or negative 6" or as "x equals plus or minus 6."

Graphing the solutions to this equation on a number line is simply putting a darkened circle on the points –6 and +6—there would be no heavy line shading because the solutions are two individual points.

FYI The graph of the solutions for absolute value equations can be shown on a real number line. Both solution values may not fit on the line unless the intervals are adjusted, as in the case of –14 and +14. Instead of numbering each mark with whole numbers, the line could have a scale of 5 with each mark labeled as –15, –10, –5, 0, +5, +10, and +15. This allows –14 and +14 to be shown as darkened circles slightly to the right of –15 and to the left of +15, respectively.

A similar absolute value equation and its solution follow:

Solve: $|c| = 14$

Solution: $c = \pm 14$

Again, the graph is two darkened circles on –14 and +14.

Now consider a more complex equation:

$$|x + 8| = 10$$

If this were written in words, it would be asking, "What values of x, when added to 8, are 10 units away from 0?" That is a rather confusing question, but the answer is easy to find with algebra.

The algebraic method relies on what you learned in the easier equation, so look at it again:

The solution of $|x| = 6$ is $x = \pm 6$, which can also be written as $x = -6$ or $x = +6$, and this is the form that helps you solve the more complex equation.

Solve: $|x| = 6$

Solution: $x = -6$ or $x = +6$

Apply this same process to $|x + 8| = 10$.

Solve: $|x + 8| = 10$

Solution: $x + 8 = -10$ or $x + 8 = +10$

Notice that the expression inside the absolute value symbol is set equal to the positive and negative values of 10. You now have two equations to solve, and each gives a solution of the original problem. The solution steps follow:

Solve: $|x + 8| = 10$

Solution: $x + 8 = -10$ or $x + 8 = +10$

Solving each equation gives the two solutions:

$x + 8 = 10$
$x + 8 - 8 = 10 - 8$
$x = 2$
or: $x + 8 = -10$
$x + 8 - 8 = -10 - 8$
$x = -18$

To check, put each value of x into the original equation:

$|x + 8| = 10 \rightarrow |2 + 8| = 10 \rightarrow |10| = 10 \rightarrow 10 = 10;$
$|x + 8| = 10 \rightarrow |-18 + 8| = 10 \rightarrow |-10| = 10 \rightarrow 10 = 10;$

Therefore, $x = 2$ and $x = -18$ are the solutions to the equation. The graph on a number line would be darkened circles at –18 and +2.

PROCEED WITH CAUTION

The checking process must be done carefully with absolute value equations. Be sure you substitute each of the two solutions into the original absolute value form because you might have made an error in the first step when you created the two new equations.

The solution to a simple equation like $|a| = 5$ is $a = \pm 5$, which can be done mentally, but if what is inside the absolute value bars is a more complex expression, always show the steps, beginning with the creation of the two equations. Another example of the more complex type is shown here:

Solve: $|d - 7| = 5$

Solution: $d - 7 = 5$ or $d - 7 = -5$

Solving each equation gives the two solutions:

$d - 7 = 5$
$d - 7 + 7 = 5 + 7$
$d = 12$
or: $d - 7 = -5$
$d - 7 + 7 = -5 + 7$
$d = 2$

To check, put each value of d into the original equation:

$|d - 7| = 5 \rightarrow |12 - 7| = 5 \rightarrow |5| = 5 \rightarrow 5 = 5;$
$|d - 7| = 5 \rightarrow |2 - 7| = 5 \rightarrow |-5| = 5 \rightarrow 5 = 5;$

Therefore, $d = 12$ and $d = 2$ are the solutions to the equation. Graphing these solutions would be darkened circles at +2 and +12.

Now try to solve these absolute value equations, checking your answers on page 290.

1. $|a| = 4$
2. $|b + 5| = 8$
3. $|x - 4| = 6$

PROCEED WITH CAUTION

Every absolute value equation has two solutions, and the more complex the variable side of the equation is, the more steps you need to show in finding the solution. Resist the temptation to do a lot of the work mentally. Solving two equations can be confusing, so show as many of the small steps as possible, to avoid careless errors.

Solving absolute value equations follows the same rules as solving any basic equation, with the important first step of writing the two new equations and then solving each one to find the two solutions. As you'll see in the next section, solving absolute value inequalities builds upon the skills you already have for solving basic inequalities.

Solving and Graphing Absolute Value Inequalities

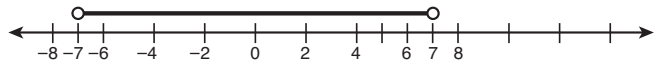
Moving on up the ladder of solving problems that include absolute value, the next step brings you to inequalities. Let's begin with a basic absolute value inequality and its solution:

$|x| < 7$

In words: "The absolute value of x is less than 7."

In other words: "What values of x are less than 7 units away from zero?"

This problem can most easily be solved with the visual help of a graph on the number line. There won't be just two points on this graph—there are many solutions to this inequality. In fact, all the numbers from 0 up to +7 and all the numbers from –7 to 0 will work. The following graph shows this solution:



Each of the numbers on the graphed line segment makes this inequality true. The circles are open on –7 and +7 because they are not included in the solution: $|-7|$ is equal to 7, not less than 7, so –7 must be excluded, and the same is true for $|+7|$. But every number—all the fractions, decimals, integers—from just above –7 up to +7 will work and are therefore on the solution graph.

Even though the graph shows you the answers to this simple inequality, the algebraic method is, of course, the better way to solve any absolute value inequality. The process is much like the one you know for solving absolute value equations, so you are really just refining those skills. And the solution will be written in algebraic form, which is far more efficient than "all the numbers between –7 and +7."

Let's solve the same absolute value inequality using algebra:

$$|x| < 7$$

If this problem was an absolute value equation, such as $|x| = 7$, you'd know the drill: the two new equations are $x = -7$ and $x = +7$, and the preferred form for the solution is $x = \pm 7$. You can do that mentally.

With an absolute value inequality, you also solve two new inequalities, but you must be careful how you write them:

One will be $x < 7$; the other will be $x > -7$.

This makes sense if you look back at the graphed solution line. But it is a process that you need to learn and use without hesitation and without having to make the graph.

FYI Explaining the process of writing the two new inequalities is much more complex than the process itself. Practice it a few times, and you'll see that it's not difficult.

The easiest way is to memorize and follow this plan:

> To create the two new inequalities, first write the problem "as it is," but without the | | symbol, and then write the problem with the inequality reversed and the sign of the number on the right side reversed.

Here's the process, worked out with an explanation:

Solve: $|x| < 7$

Solution: $x < 7$ and $x > -7$ are the new inequalities. Because both of these inequalities must be true at the same time, the solution is written in interval form, based on combining both inequalities:

> $x > -7$ can be written as $-7 < x$. Because $x < +7$, these both combine to make the interval $-7 < x < +7$, and this is the final solution.

Looking at the graph of this absolute value inequality, the interval solution $-7 < x < +7$ exactly matches the graph and can be read as "–7 is less than x, which is less than +7."

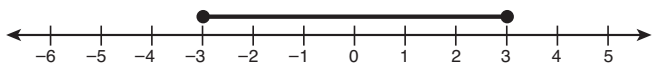
The following example includes only the solution steps:

> $|a| \leq 3$

PROCEED WITH CAUTION

Be careful as you write the new inequalities—especially the second one, with its reversed inequality symbol and its reversed sign on the number on the right side of the inequality. The wrong sign or symbol will lead to a wrong solution.

Solution: $a \leq 3$ and $a \geq -3$ are the new inequalities; the combined inequality solution is $-3 \leq a \leq 3$ and the graph follows:



When the inequality includes a more complex expression within the absolute value symbol, the new inequalities are created with the same process. Solving these new problems just takes a little more work, but it is very similar to the inequality work you did in Hour 19. Some examples follow:

$|x - 3| < 5$

Solution: $x - 3 < 5$ and $x - 3 > -5$ are the new inequalities, solved in the following:

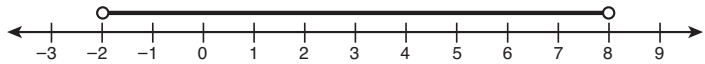
$x - 3 < 5$
$x - 3 + 3 < 5 + 3$
$x < 8$
and
$x - 3 > -5$
$x - 3 + 3 > -5 + 3$
$x > -2$

Combining the two solutions, $-2 < x < 8$.

JUST A MINUTE

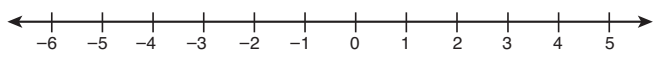
Remember that an open circle on the endpoint of a graph means that the number is not included in the solution. A darkened circle means it is a part of the solution, along with the rest of the darkened line: < and > have open circles; ≤ and ≥ have darkened circles.

The following is the graph:

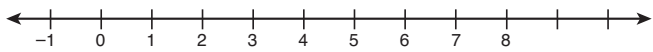


Now try to solve these absolute value inequalities. Graph their solutions, and check your answers on page 290.

4. $|x| \leq 5$



5. $|a - 4| < 3$

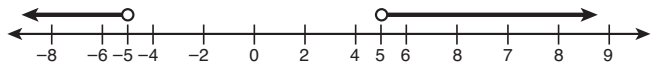


All the absolute value inequalities so far have included either < or ≤ as their relation. A variation in the final solution process comes into play when the > and ≥ inequalities are used, as you'll see in the next examples.

Beginning with a basic inequality, you use the same process to write the two new inequalities:

$$|x| > 5$$

Solution: $x > 5$ or $x < -5$ are the new inequalities, and they need no further work. However, when you graph these solutions, something looks different:



FYI The solution of > and ≥ inequalities have two parts that are both inequalities, and they are most easily seen if you graph them on a number line.

The solution has two parts on the graph, and between these is open space. Choose a point in that open space—let's use 2—and put it into the original inequality:

$$|2| > 5 \rightarrow 2 > 5 \rightarrow \text{False}$$

This shows that all the numbers in the open space are not solutions of the inequality. If you put any number from either of the graphed lines into the original problem, you will get a true statement because those lines do represent the solution to this inequality.

The following slightly more complex inequality will have a similar solution that can be graphed with two separate lines.

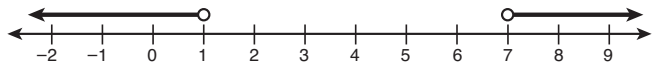
$$|x - 4| > 3$$

Solution: The new inequalities are created as before, and for this problem, they are $x - 4 > 3$ or $x - 4 < -3$. The solution steps follow:

$x - 4 > 3$
$x - 4 + 4 > 3 + 4$
$x > 7$
or
$x - 4 < -3$
$x - 4 + 4 < -3 + 4$
$x < 1$

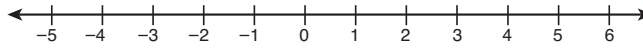
When the solutions are graphed, the lines form two separate sets of numbers that are solutions of the original inequality:

FYI The open space between the two graphed lines shows the numbers that are not solutions of the original inequality.

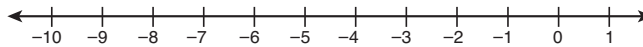


Now try to solve these absolute value inequalities. Graph their solutions, and check your answers on page 291.

6. $|x| \geq 2$



7. $|b + 4| > 4$



Your ability to solve equations and inequalities that include absolute value expressions pushes your algebra skills to an advanced level as it expands your mind's ability to process increasingly complex ideas. Just as this hour has built upon your equation-solving abilities, the next section exposes you to new ways of looking at graphing lines on the (x,y) coordinate axes that you first saw in Hour 15.

Graphing Inequalities with Two Variables

The graphs of single-variable equation solutions can be drawn on a number line, as you have been doing for the past few hours. When equations have two variables—typically x and y—that are part of a general type of equation like $y = mx + b$, their solutions can be graphed as ordered pairs and the line that connects these ordered pairs.

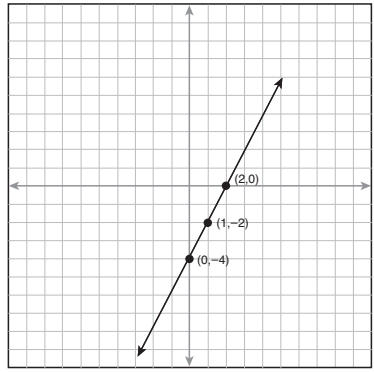
For the final part of this hour, let's look at solving some two-variable inequalities, such as the following:

$$y \leq 2x - 4$$

In the same way you began when learning how to solve other inequalities, look at this problem written instead as an equation:

$$y = 2x - 4$$

It is in the slope-intercept form of $y = mx + b$, where m is the slope and b is the y-intercept. The slope of this line is 2, and the y-intercept is -4. The following graph on the x- and y-axes shows this equation's line:



Every point on this line is a solution of the equation $y = 2x - 4$, and there are an infinite number of solutions because there are an infinite number of points on the line. This review of graphing the solution of an equation helps you see all those solutions (ordered pairs) more easily.

GO TO ◀

Recall that $y = mx + b$ is the slope-intercept form of a line that has a slope m and a y-intercept b. The line $y = 2x + 6$ has a slope—the ratio of rise to run—of 2 and a y-intercept of 6. The intercepts are (0,6) and (–3,0). See Hour 15 for a review of graphing lines.

Now return to the inequality $y \leq 2x - 4$.

Its graph is based on the graph of the line $y = 2x - 4$. However, the inequality includes even more points than are on the line. Your goal is to find all the (x,y) values that make this inequality a true statement. Because $\leq$ means "less than or equal to," you know that the line shows all the "equal to" points.

Where are the "less than" ordered pairs?

The best way to find them is to choose a specific point that is not on the line and check it in the original inequality. If it makes a true statement, all the points in that region of the graph will make a true statement.

Let's use the ordered pair (5,0). It lies to the right of the line. Does it make a true statement? Put 5 in for x and 0 in for y in the original inequality:

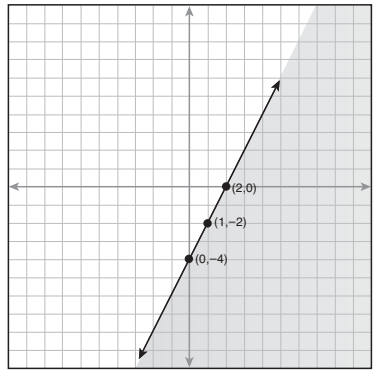
$$y \leq 2x - 4 \rightarrow 0 \leq 2 \cdot 5 - 4 \rightarrow 0 \leq 10 - 4 \rightarrow 0 \leq 6 \rightarrow \text{True}$$

FYI If an ordered pair from one side of the graphed line makes a true statement when put into the original inequality, then all the ordered pairs on that side of the line will also make a true statement. They are all solutions of the inequality.

Yes, the ordered pair (5,0) is a solution for this inequality. All the ordered pairs to the right of the line will make a true statement when put into the original inequality. None of the ordered pairs on the left side will make a true statement. Let's confirm this using a chosen point, (0,3). Put 0 in for x and 3 in for y in the original inequality:

$$y \leq 2x - 4 \rightarrow 3 \leq 2 \cdot 0 - 4 \rightarrow 3 \leq 0 - 4 \rightarrow 3 \leq -4 \rightarrow \text{False}$$

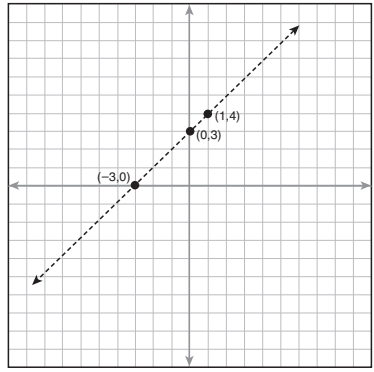
To show the graph of all the ordered pairs that are solutions of the inequality, you shade in the area of the graph lightly to the right of and below the graphed line, as shown in the following:



The final example shows the solution of another inequality with two variables:

$$y > x + 3$$

First, graph the line from the equation $y = x + 3$. This line has a slope of 1 and y-intercept of 3, with key points (0,3), (1,4), and (–3,0); the graph is shown here:



FYI In the graphs on the real number line for inequalities such as $x > 3$, the open circle on +3 shows that 3 is not a part of the solution, but the heavy line extending to the right of +3 shows all the numbers that are the solution. In the same way, lines on (x,y) graphs are dashed, not solid, if the inequality is < or >. This shows that the ordered pairs on the line are not part of the solution, but the shaded area is the solution.

Notice that the line is not drawn as a solid line, but is dashed. This is because the original inequality was >, not ≥, so the ordered pairs on this line are not actually part of the solution.

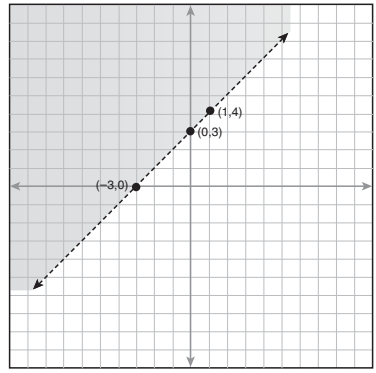
Now choose a point to the right of the line, such as (0,2), and put 0 in for x and 2 in for y in the original inequality:

$$y > x + 3 \rightarrow 2 > 0 + 3 \rightarrow 2 > 3 \rightarrow \text{False}$$

This shows that all the ordered pairs on the right side of the line will make false statements, so the shading should go on the left side. To confirm this, choose an ordered pair on the left, such as (–2,4), and put –2 in for x and 4 in for y in the original inequality:

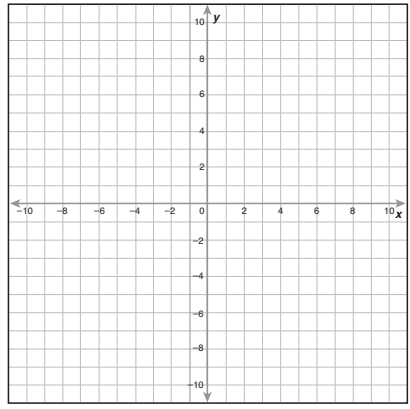
$$4 > -2 + 3 \rightarrow 4 > 1 \rightarrow \text{True}$$

This confirms that all the ordered pairs to the left of and above the dashed line are solutions, so that region should be shaded, as follows:

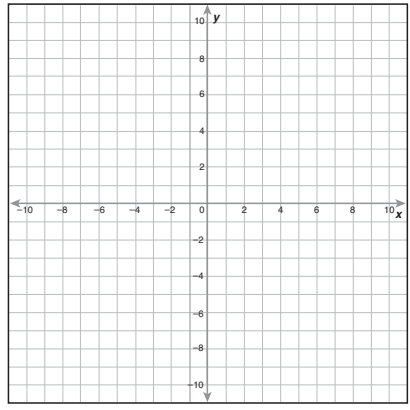


Now try to graph the solution of the following inequalities, checking your answers on pages 291–292.

8. $y < 3x - 6$; slope is 3 and y-intercept is -6



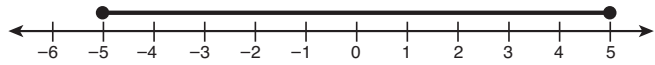
9. $y \geq 2x + 6$; slope is 2 and y-intercept is 6



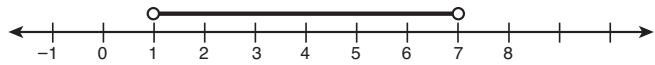
Graphing the lines at the heart of two-variable inequalities gives you the boundary that separates the two regions of the (x,y) ordered pairs; one area has all the solutions, and the other area has none. This allows you to easily identify whether any specific ordered pair is a solution of the inequality.

Answers to Sample Problems

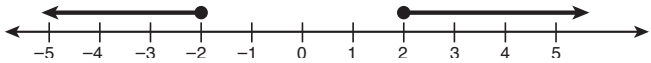
1. $|a| = 4 \rightarrow a = \pm 4$
2. $|b + 5| = 8 \rightarrow b + 5 = 8$ or $b + 5 = -8$
   $b + 5 = 8$
   $b + 5 - 5 = 8 - 5$
   $b = 3$
   $b + 5 = -8$
   $b + 5 - 5 = -8 - 5$
   $b = -13$
   $b = -13$ or 3
3. $|x - 4| = 6 \rightarrow x - 4 = 6$ or $x - 4 = -6$
   $x - 4 + 4 = 6 + 4$
   $x = 10$
   $x - 4 + 4 = -6 + 4$
   $x = -2$
   $x = -2$ or 10
4. $|x| \leq 5 \rightarrow x \leq 5$ and $x \geq -5 \rightarrow$ combine to give the interval $-5 \leq x \leq 5$



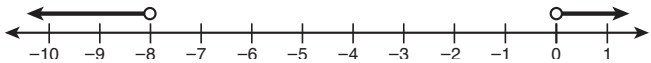
5. $|a - 4| < 3 \rightarrow a - 4 < 3$ and $a - 4 > -3$
   $a - 4 + 4 < 3 + 4$
   $a < 7$
   $a - 4 + 4 > -3 + 4$
   $a > 1$
   $a > 1$ and $a < 7$ combine to give the interval $1 < a < 7$



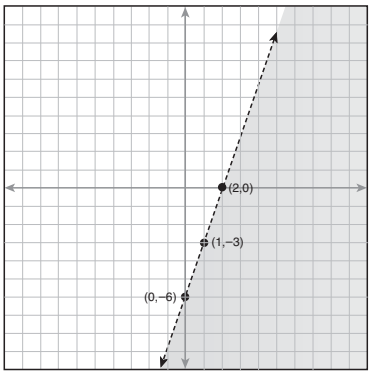
6. $|x| \geq 2 \rightarrow x \geq 2$ or $x \leq -2$



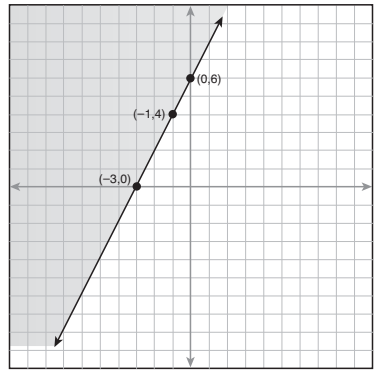
7. $|b + 4| > 4 \rightarrow b + 4 > 4$ or $b + 4 < -4$
   $b + 4 - 4 > 4 - 4$
   $b > 0$
   $b + 4 - 4 < -4 - 4$
   $b < -8$



8. $y < 3x - 6$; slope is 3 and y-intercept is -6



9. $y \geq 2x + 6$; slope is 2 and y-intercept is 6



Review

Hour's Up!

Based on what you've learned in Hour 20, find the solutions for the following absolute value equations and inequalities. Check your answers with the solution key in Appendix B.

1. $|x| = 9$
- **a.** $x = 9$
- **b.** $x = -9$
- **c.** $x = \pm 9$
- **d.** $x < 9$

2. $|c - 6| = 7$
- **a.** $c = 1$ or -13
- **b.** $c = -1$ or 13
- **c.** $c = 1$ or 13
- **d.** $c = -1$ or -13

3. $|x + 2| = 8$
- **a.** $x = -10$ or -6
- **b.** $x = 10$ or 6
- **c.** $x = 10$ or -6
- **d.** $x = -10$ or 6

4. $|x| < 4$
a. $x < 4$
b. $x < -4$
c. $-4 < x < 4$
d. $4 < x < -4$

5. $|a + 7| \leq 3$
a. $4 \leq a \leq -10$
b. $-10 \leq a \leq -4$
c. $a \leq 10$
d. $a \leq -10$

6. $|x - 3| \leq 1$
a. $2 \leq x \leq 4$
b. $-2 \leq x \leq 4$
c. $2 \leq x \leq -4$
d. $-2 \leq x \leq 4$

7. $|a - 2| < 7$
a. $5 < a < 9$
b. $-5 < a < -9$
c. $-5 < a < 9$
d. $5 < a < -9$

8. $|c| > 3$
a. $c < -3$ or $c > 3$
b. $c < 3$ or $c > -3$
c. $-3 < c < 3$
d. $c < -3$

9. $|x - 8| \geq 6$
a. $x \leq -2$ or $x \geq 14$
b. $2 \leq x \leq 14$
c. $x \geq 2$ or $x \leq 14$
d. $x \leq 2$ or $x \geq 14$

10. $|x + 2| > 4$
a. $-6 < x < 2$
b. $x < 6$ or $x > 2$
c. $x < -6$ or $x > 2$
d. $x < -2$ or $x > 6$

Review

Part VI

Quadratic Equations

Hour 21 Square Roots: Simplifying and Solving Equations

Hour 22 Square Roots: Four Operations and Radical Equations

Hour 23 Quadratic Equations: Solving by Three Methods

Hour 24 Quadratic Equations: Solving by the Quadratic Formula

Hour 21

Square Roots: Simplifying and Solving Equations

Chapter Summary

In this hour you will learn about …

- Simplifying square roots of numbers
- Simplifying square roots that include variables
- Solving equations that include a squared variable

As you begin this fourth and final part of studying algebra, it's appropriate that you'll be learning about quadratic equations. The word *quadratic* is based on *quadratum*, the Latin word for "square," a shape with four equal sides. If the length of each side is x, the area is x times x, x^2, or "x squared." The tie-in is that every quadratic equation has a variable—typically an x—that is squared, and there are several ways to find the solution. You will also see the connection between algebra and the real world in formulas from research, finance, and geometric shapes. Focusing briefly on a square shape, the length of each side of that square, and the square's area leads to the main topic for the next two hours.

Simplifying Square Roots of Numbers

Visualize the following square shape: it has four sides, all equal in length, and four right angles. If each side of this square is 3 inches long, you could draw 9 little 1-inch squares within this 3-by-3 square. Make a sketch to see it more clearly. If each side of another square is 4 inches long, there will be 16 little 1-inch squares inside it. So the area of a square can be found by drawing the square and counting the little squares inside it, or you could just multiply the length of a side times itself, which is much more efficient.

Based on each of these squares, you know that a 3-by-3-inch square contains 9 square inches, and a 4-by-4-inch square contains 16 square inches. Rather than drawing and counting all the squares, the algebraic shortcut lets you write it this way:

In words		In algebra
3 squared is 9	→	$3^2 = 9$
4 squared is 16	→	$4^2 = 16$

You are familiar with the four basic operations of adding, subtracting, multiplying, and dividing. Now a fifth operation is added to your algebra toolbox: squaring. It's based on the area of an actual square shape, but its applications are almost endless.

Formulas of all types include variables that are squared, and general equations that have squared variables are so common, they have their own name—quadratic equations. As with solving absolute value equations, finding the solutions of quadratic equations is based on all the equation-solving skills you already have, pushed up to a slightly higher level, but it's a step-by-step process that can be learned easily. You have to learn some basic work with squared numbers and variables first.

In much the same way that subtraction is the opposite operation of addition, $2 + 3 = 5 \rightarrow 5 - 3 = 2$, the opposite operation of squaring is called the square root. The following examples show how to find the square root of a number:

FYI The number inside the square root symbol comes from multiplying a number times itself—a number squared. Your job is to find the numbers that, when squared, give the number inside the square root symbol.

Because $3^2 = 9$, the square root of 9 is 3. The algebraic symbol for square root is $\sqrt{\ }$ (often called a radical sign or simply radical), and this example can be written as follows:

$$3^2 = 9 \rightarrow \sqrt{9} = 3$$

It can be spoken as, "The square root of 9 is 3 because 3 times 3 is 9."

Similarly, $4^2 = 16 \rightarrow \sqrt{16} = 4$

"The square root of 16 is 4 because 4 times 4 is 16."

However, other numbers multiplied times themselves give 9 and 16: from your work with negative numbers, you know that $-3 \cdot -3 = 9$ and $-4 \cdot -4 = 16$, or $(-3)^2 = 9$ and $(-4)^2 = 16$.

These examples can be written as follows:

$(-3)^2 = 9 \rightarrow \sqrt{9} = -3$ because –3 times –3 is 9.
$(-4)^2 = 16 \rightarrow \sqrt{16} = -4$ because –4 times –4 is 16.

Because there are two ways to get a product of 9 by multiplying a number times itself—$3 \cdot 3$ and $-3 \cdot -3$—there are two answers to $\sqrt{9}$.

$$\sqrt{9} = 3 \text{ or } \sqrt{9} = -3$$

The most efficient way to write these as a single expression is to use the ± symbol:

$$\sqrt{9} = \pm 3 \rightarrow \text{"The square root of 9 is positive or negative 3."}$$

Similarly, $\sqrt{16} = \pm 4 \rightarrow$ "The square root of 16 is positive or negative 4."

In common usage, only the positive answer is shown, and it is called the principal square root. Most formula applications that include the square root symbol require only the positive answer, so generally you will see $\sqrt{9} = 3$. If both answers are needed, the problem is usually written as $\pm\sqrt{9} = \pm 3$.

What is $\sqrt{0}$? You know that $0 \cdot 0 = 0$, so $\sqrt{0} = 0$, without the ± because 0 isn't positive or negative. Keep in mind that the square root of any positive number does have two possible answers, if both are needed.

FYI What about the square root of a negative number? What is the square root of –9? Because no number times itself gives a negative product (a positive times a positive gives a positive, and a negative times a negative also gives a positive), no real number can be the square root of –9. Square roots of negative numbers are called imaginary numbers, which is a topic for higher-level algebra courses and is included in Hour 24.

Because square roots are based on squared numbers, it is helpful for you to memorize the following:

$0^2 = 0 \rightarrow \sqrt{0} = 0$
$1^2 = 1 \rightarrow \sqrt{1} = 1$
$2^2 = 4 \rightarrow \sqrt{4} = 2$
$3^2 = 9 \rightarrow \sqrt{9} = 3$
$4^2 = 16 \rightarrow \sqrt{16} = 4$
$5^2 = 25 \rightarrow \sqrt{25} = 5$
$6^2 = 36 \rightarrow \sqrt{36} = 6$
$7^2 = 49 \rightarrow \sqrt{49} = 7$
$8^2 = 64 \rightarrow \sqrt{64} = 8$
$9^2 = 81 \rightarrow \sqrt{81} = 9$
$10^2 = 100 \rightarrow \sqrt{100} = 10$

Working with square root problems is much easier if you recognize the "perfect squares" from the list; they are shown here:

0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

When you take the square root of a perfect square, the answer is a whole number. In the same way that knowing the basic multiplication facts for the numbers 0 through 10 lets you focus on more complex algebra problems, knowing that a number is a perfect square saves valuable time and energy when you are simplifying square root problems.

Consider the following examples of finding square roots:

$\sqrt{49} = 7$ (49 is the perfect square of 7)
$\sqrt{81} = 9$ (81 is the perfect square of 9)

These two problems are easy, since 49 and 81 are perfect squares.

FYI It will be helpful to factor the number inside the square root symbol if it isn't on the perfect square list. Remember that factoring is simply writing a number as the product of two or more numbers: examples include $900 = 9 \cdot 100$ and $144 = 4 \times 36$. Use factors that are perfect squares whenever possible: $144 = 2 \cdot 72$ is true, but the perfect square factors $4 \cdot 36$ allow you to find the square root of 144.

$\sqrt{400}$ is not so easy, because 400 isn't on your list of perfect squares. Your factoring skills and the list of perfect squares help you work around this problem:

Think about 400. Are there any numbers in the perfect square list that are factors of 400? Yes: 4 and 100.

$$\sqrt{400} = \sqrt{4 \cdot 100}$$

This factoring process lets you break down 400 into manageable pieces. A square root problem can always be written as the product of two (or more) square roots. This means that the factors of 4 and 100 can be separated into two multiplied square roots:

$$\sqrt{400} = \sqrt{4 \cdot 100} = \sqrt{4} \cdot \sqrt{100}$$

You know that $\sqrt{4} = 2$ and $\sqrt{100} = 10$. The simplified answer follows:

$$\sqrt{400} = \sqrt{4 \cdot 100} = \sqrt{4} \cdot \sqrt{100} = 2 \cdot 10 = 20$$

You might have been able to find $\sqrt{400} = 20$ mentally, but the algebra steps of factoring 400 and using the product of the two square roots gets you the answer without a lot of work. If you show only the first step of factoring and keep the numbers inside the extended square root symbol, that is fine, too.

The square root of a fraction can be simplified in a very similar way, as shown here:

$\sqrt{\frac{16}{25}}$ can be written as $\frac{\sqrt{16}}{\sqrt{25}}$ because the square root of a quotient is equal to the quotient of the square roots.

$$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$$

FYI The number inside the square root symbol is the radicand. The square root symbol itself is often called the radical. You may hear the phrase "radical 16," which is another way to say "the square root of 16."

Now try to simplify the following square root problems, checking your answers on page 310.

1. $\sqrt{64} =$
2. $\sqrt{2500} =$
3. $\sqrt{\frac{81}{49}} =$

The square root problems so far have all included radicands that are perfect squares or perfect square factors. Their simplified answers have no square root symbols because the square root of a perfect square will always be a whole number. Even the examples that contained a fraction simplified to have whole number numerators and denominators.

Consider the following radical:

Simplify: $\sqrt{20}$

Because 20 is not a perfect square, try factoring it into two numbers. The possible factors are shown here:

$$\sqrt{20} = \sqrt{4 \cdot 5} \text{ or } \sqrt{20} = \sqrt{2 \cdot 10}$$

The problem is that, in the first solution, 5 is not a perfect square, and, in the second solution, neither 2 nor 10 is a perfect square.

When this happens, choose the solution that has at least one perfect square factor and simplify it as much as possible:

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$$

FYI The order of the factors inside the square root symbol does not affect the final answer. The perfect square factor will simplify to a whole number, regardless of whether it is the first or second factor.

A square root expression is in simplest form when all the perfect square factors have been changed to their whole number answers, so this problem is finished. It can be read as "The square root of 20 is two times the square root of five," or as "The square root of 20 is two root five."

Knowing all the numbers that are perfect squares from 4 to 100 will certainly help you simplify square root problems, as shown in the following examples:

Simplify: $\sqrt{50}$

The factors of 50 are $5 \cdot 10$ and $2 \cdot 25$. Use the second pair, because 25 is a perfect square—5 and 10 are not perfect squares.

$$\sqrt{50} = \sqrt{2 \cdot 25} = 5\sqrt{2}$$

Note that the two earlier examples had similar answers, but $2\sqrt{5}$ is very different from $5\sqrt{2}$—the first is $\sqrt{20}$, the second is $\sqrt{50}$.

In fraction problems, simplifying the fraction first, if possible, usually makes the problem easier. By factoring out the common factor from both the numerator and denominator, the remaining numbers might be perfect squares, as shown here:

$$\sqrt{\frac{2}{32}} = \sqrt{\frac{2 \cdot 1}{2 \cdot 16}} = \sqrt{\frac{1}{16}} = \frac{\sqrt{1}}{\sqrt{16}} = \frac{1}{4}$$

$$\sqrt{\frac{24}{54}} = \sqrt{\frac{6 \cdot 4}{6 \cdot 9}} = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

FYI All fraction radicands do not simplify to create perfect square numerators and denominators. That type of problem is presented in higher-lever algebra courses.

Now try to simplify the following square root problems, checking your answers on page 310.

4. $\sqrt{28} =$
5. $\sqrt{75} =$
6. $\sqrt{\frac{20}{80}} =$
7. $\sqrt{\frac{12}{27}} =$

When a radical expression is a combination of a whole number multiplied by a square root, the process of simplifying is a bit more complex.

Consider the following problem:

Simplify: $8\sqrt{36}$

Because this problem is just "8 times the square root of 36" or "8 root 36," simplify the radical and multiply the result by 8:

$$8\sqrt{36} = 8 \cdot 6 = 48$$

The perfect square radicand made that problem an easy one.

Simplify: $3\sqrt{32}$

Find the perfect square factors of 32, if there are any:

$$3\sqrt{32} = 3 \cdot \sqrt{16 \cdot 2} = 3 \cdot \sqrt{16} \cdot \sqrt{2} = 3 \cdot 4 \cdot \sqrt{2} = 12\sqrt{2}$$

An alternative method is shown here:

$$3\sqrt{32} = 3 \cdot \sqrt{4 \cdot 8} = 3 \cdot \sqrt{4} \cdot \sqrt{8} = 3 \cdot 2 \cdot \sqrt{8} = 6\sqrt{8}$$

However, because $2 \cdot 4 = 8$, you need to continue to simplify the problem:

$$6\sqrt{8} = 6 \cdot \sqrt{4 \cdot 2} = 6 \cdot \sqrt{4} \cdot \sqrt{2} = 6 \cdot 2 \cdot \sqrt{2} = 12\sqrt{2}$$

The previous example shows clearly how important it is to factor the radicand into the largest perfect square factor possible. It saves a lot of time and potential errors.

When 16 and 2 were used for 32, the process took half as many steps as when 4 and 8 were the factors.

Now try to simplify the following square root problems, checking your answers on page 310.

8. $5\sqrt{81} =$
9. $12\sqrt{40} =$

A similar process is used when variables are inside the square root symbol, with a variation that helps you simplify the terms with exponents. The number part of each expression is simplified as in all the problems so far. This will prepare you for solving the equations at the end of the hour.

Simplifying Square Roots That Include Variables

The final simplification process applies to square root problems that have variables in the radicand, as in the following examples:

$$\sqrt{25x^2}$$

The radicand can be separated into the product of two radicals and then simplified:

$$\sqrt{25x^2} = \sqrt{25} \cdot \sqrt{x^2} = 5\sqrt{x^2}$$

PROCEED WITH CAUTION

Remember that your job in simplifying radicals is to find the numbers (or variables) that, when squared, give the number (or variables) inside the square root symbol. Pay close attention to the exponents of the variables in the square root.

$\sqrt{25x^2} = \sqrt{25} \cdot \sqrt{x^2} = 5\sqrt{x^2}$ Is $\sqrt{x^2}$ in simplest terms? No, because the x^2 term inside the square root symbol is $x \cdot x$, so $\sqrt{x^2}$ must be x. Continuing to simplify gives this:

$$5\sqrt{x^2} = 5 \cdot x = 5x$$

With more complex radicands with variables, use the same steps for simplifying nonvariable problems, paying attention to how the numbers and the variables are affected by the square root symbol.

Simplify: $3\sqrt{12x^{10}}$

Work on finding the perfect square factors of 12, if there are any, and on finding the square root of the variable. The square root term will move to the end of the simplified terms:

$$3\sqrt{12x^{10}} = 3 \cdot \sqrt{4 \cdot 3} \cdot \sqrt{x^5 \cdot x^5} = 3 \cdot 2 \cdot \sqrt{3} \cdot x^5 = 6x^5\sqrt{3}$$

Simplify: $\sqrt{x^7}$

Factor the x term into clusters, which for an exponent of 7 is shown here:

$$\sqrt{x^7} = \sqrt{x^6 \cdot x} = \sqrt{x^3 \cdot x^3 \cdot x} = x^3 \cdot \sqrt{x} = x^3\sqrt{x}$$

Because $x^3 \cdot x^3 = x^6$, $\sqrt{x^6} = x^3$. The $\sqrt{x}$ term cannot be simplified and stays at the end as a multiplier.

Simplify: $5\sqrt{75x^3}$

In this problem, the exponent on the x term is a 3, so the factors of the variable term look like this:

$$5\sqrt{75x^3} = 5 \cdot \sqrt{25 \cdot 5 \cdot x^2 \cdot x} = 5 \cdot \sqrt{25} \cdot \sqrt{5} \cdot \sqrt{x^2} \cdot \sqrt{x} = 5 \cdot 5 \cdot \sqrt{5} \cdot x \cdot \sqrt{x} = 25x\sqrt{5x}$$

Simplify: $7\sqrt{12x^8y^5z}$

$$7\sqrt{12x^8y^5z} = 7 \cdot \sqrt{4 \cdot 3 \cdot x^4 \cdot x^4 \cdot y^2 \cdot y^2 \cdot y \cdot z} = 7 \cdot 2 \cdot x^4 \cdot y^2 \cdot \sqrt{3yz} = 14x^4y^2\sqrt{3yz}$$

PROCEED WITH CAUTION

When variables with the same base are multiplied, the exponents are added. Be careful to not take the square root of the exponent. Cluster the variables with exponents so you can see the terms multiplied by each other. These terms move out of the radical to become a single term, in much the same way that perfect squares move out.

Now try to simplify the following square root problems, checking your answers on page 310.

10. $10\sqrt{9a^2b^8c^7} =$
11. $2\sqrt{40x^6y^3} =$

Your skills in simplifying square roots make it possible to solve a simple quadratic equation, as you will see in this hour's final section.

Solving Equations That Include a Squared Variable

If an equation has a squared variable, isolating that variable requires something other than the properties of addition, subtraction, multiplication, or division. The only way to isolate a squared variable is to take its square root, and that is exactly how you can solve a basic quadratic equation.

Consider the following quadratic equation:

Solve: $x^2 = 25$

Solution: None of the basic operations will help you find the value of x that will make this a true statement, which is your goal in solving this equation. If a variable x is squared, taking its square root will equal x. By taking the square root of both sides of this equation, the variable will be isolated. The solution steps follow:

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm 5$$

FYI If you take the square root of a number, both the positive and negative answers are the solution, because squaring each of these gives the number in the radicand. Check both solutions in the original equation.

To check, put +5 and –5 in for x in the original equation:

$$x^2 = 25 \rightarrow (+5)^2 = 25 \rightarrow 25 = 25 \text{ and}$$
$$x^2 = 25 \rightarrow (-5)^2 = 25 \rightarrow 25 = 25$$

Therefore, both +5 and –5 are the solutions to the equation $x^2 = 25$.

You need to isolate the squared variable term before taking the square root of both sides of the equation, as in this example:

$$x^2 - 36 = 0$$
$$x^2 - 36 + 36 = 0 + 36$$
$$x^2 = 36$$
$$\sqrt{x^2} = \sqrt{36}$$
$$x = \pm 6$$

To check, put +6 and –6 in for x in the original equation:

$$x^2 - 36 = 0 \rightarrow (+6)^2 - 36 = 0 \rightarrow 36 - 36 = 0 \rightarrow 0 = 0$$
and
$$x^2 - 36 = 0 \rightarrow (-6)^2 - 36 = 0 \rightarrow 36 - 36 = 0 \rightarrow 0 = 0$$

Therefore, both +6 and –6 are the solutions to the equation $x^2 - 36 = 0$.

If the squared variable term has a coefficient other than 1, use the division property to isolate the squared variable, and then take the square root of both sides of the equation:

$$2x^2 = 18$$
$$\frac{2x^2}{2} = \frac{18}{2}$$
$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = \pm 3$$

To check, put +3 and –3 in for x in the original equation:

$2x^2 = 18 \rightarrow 2 \cdot (+3)^2 = 18 \rightarrow 2 \cdot 9 = 18 \rightarrow 18 = 18$
and
$2x^2 = 18 \rightarrow 2 \cdot (-3)^2 = 18 \rightarrow 2 \cdot 9 = 18 \rightarrow 18 = 18$

Therefore, both +3 and –3 are the solutions to the equation $2x^2 = 18$.

Similarly, isolating the squared variable term in the next example allows you to find the solutions by taking the square root of both sides of the equation:

$$36x^2 = 49$$
$$36x^2 \div 36 = 49 \div 36$$
$$x^2 = \frac{49}{36}$$
$$\sqrt{x^2} = \sqrt{\frac{49}{36}}$$
$$x = \pm\frac{7}{6}$$

To check, put $+\frac{7}{6}$ and $-\frac{7}{6}$ in for x in the original equation:

$$36x^2 = 49 \rightarrow 36 \cdot \left(+\frac{7}{6}\right)^2 = 49 \rightarrow 36 \cdot \frac{49}{36} = 49 \rightarrow 49 = 49$$
and
$$36x^2 = 49 \rightarrow 36 \cdot \left(-\frac{7}{6}\right)^2 = 49 \rightarrow 36 \cdot \frac{49}{36} = 49 \rightarrow 49 = 49$$

Therefore, both $+\frac{7}{6}$ and $-\frac{7}{6}$ are the solutions to the equation $36x^2 = 49$.

The next example shows the solution steps for a quadratic equation with a coefficient other than 1, along with a number subtracted from the variable term. Moving the subtracted term and then isolating the squared

variable allows you to take the square root of both sides and find the solutions:

$$\begin{aligned} 25x^2 - 81 &= 0 \\ 25x^2 - 81 + 81 &= 0 + 81 \\ 25x^2 &= 81 \\ \frac{25x^2}{25} &= \frac{81}{25} \\ x^2 &= \frac{81}{25} \\ \sqrt{x^2} &= \sqrt{\frac{81}{25}} \\ x &= \pm\frac{9}{5} \end{aligned}$$

To check, put $+\frac{9}{5}$ and $-\frac{9}{5}$ in for x in the original equation:

$25x^2 - 81 = 0 \rightarrow 25 \cdot \left(+\frac{9}{5}\right)^2 - 81 = 0 \rightarrow 25 \cdot \frac{81}{25} - 81 = 0 \rightarrow$
$81 - 81 = 0 \rightarrow 0 = 0$

and

$25x^2 - 81 = 0 \rightarrow 25 \cdot \left(-\frac{9}{5}\right)^2 - 81 = 0 \rightarrow 25 \cdot \frac{81}{25} - 81 = 0 \rightarrow$
$81 - 81 = 0 \rightarrow 0 = 0$

Therefore, both $+\frac{9}{5}$ and $-\frac{9}{5}$ are the solutions to the equation $25x^2 - 81 = 0$.

Although the last two equations were more complicated than the earlier examples, these quadratic equations are considered basic because they all use the same solving techniques as nonquadratic equations. The only difference is at the very end of the solution steps, when you must take the square root of both sides to get the final solutions.

Now try to solve this quadratic equation, checking your answer on page 310.

12. $3x^2 - 20 = 28$

The equations have become more sophisticated, but your strong algebra skills will help you move successfully to the next hour's quadratic equations and new solution methods.

Answers to Sample Problems

1. 8
2. $\sqrt{2500} = \sqrt{25 \cdot 100} = \sqrt{25} \cdot \sqrt{100} = 5 \cdot 10 = 50$
3. $\sqrt{\frac{81}{49}} = \frac{\sqrt{81}}{\sqrt{49}} = \frac{9}{7}$
4. $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4} \cdot \sqrt{7} = 2\sqrt{7}$
5. $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$
6. $\sqrt{\frac{20}{80}} = \sqrt{\frac{20 \cdot 1}{20 \cdot 4}} = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$
7. $\sqrt{\frac{12}{27}} = \sqrt{\frac{3 \cdot 4}{3 \cdot 9}} = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
8. $5\sqrt{81} = 5 \cdot 9 = 45$
9. $12\sqrt{40} = 12 \cdot \sqrt{4 \cdot 10} = 12 \cdot \sqrt{4} \cdot \sqrt{10} = 12 \cdot 2 \cdot \sqrt{10} = 24\sqrt{10}$
10. $10\sqrt{9a^2b^8c^7} = 10 \cdot \sqrt{9 \cdot a^2 \cdot b^4 \cdot b^4 \cdot c^3 \cdot c^3 \cdot c} = 10 \cdot 3 \cdot a \cdot b^4 \cdot c^3 \cdot \sqrt{c} = 30ab^4c^3\sqrt{c}$
11. $2\sqrt{40x^6y^3} = 2 \cdot \sqrt{4 \cdot 10 \cdot x^3 \cdot x^3 \cdot y^2 \cdot y} = 2 \cdot 2 \cdot x^3 \cdot y \cdot \sqrt{10 \cdot y} = 4x^3y\sqrt{10y}$
12. $3x^2 - 20 = 28$

$3x^2 - 20 + 20 = 28 + 20$

$3x^2 = 48$

$\frac{3x^2}{3} = \frac{48}{3}$

$x^2 = 16$

$\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

Hour's Up!

Review

Based on what you've learned in Hour 21, simplify the following square root expressions. Check your answers with the solution key in Appendix B.

1. $\sqrt{3600} =$
   - **a.** 60
   - **b.** 600
   - **c.** 10
   - **d.** 6
2. $\sqrt{\frac{25}{64}} =$
   - **a.** $\frac{8}{5}$
   - **b.** $\frac{5}{8}$
   - **c.** $\frac{5}{9}$
   - **d.** $\frac{9}{5}$
3. $\sqrt{40} =$
   - **a.** $2\sqrt{20}$
   - **b.** $4\sqrt{10}$
   - **c.** $10\sqrt{2}$
   - **d.** $2\sqrt{10}$
4. $\sqrt{45} =$
   - **a.** $5\sqrt{9}$
   - **b.** $9\sqrt{5}$
   - **c.** $5\sqrt{3}$
   - **d.** $3\sqrt{5}$
5. $\sqrt{\frac{10}{90}} =$
   - **a.** $\frac{1}{9}$
   - **b.** 3
   - **c.** $\frac{1}{3}$
   - **d.** 9

6. $\sqrt{\frac{50}{32}} =$

a. $\frac{5}{4}$

b. $\frac{4}{5}$

c. $\frac{25}{16}$

d. $\frac{5}{16}$

7. $3\sqrt{25} =$

a. 75

b. 15

c. 8

d. 5

8. $6\sqrt{20} =$

a. 30

b. 120

c. $12\sqrt{5}$

d. $5\sqrt{12}$

9. $5\sqrt{16ab^2c^5} =$

a. $4bc^2\sqrt{ac}$

b. $20ab^2c^5\sqrt{5}$

c. $4abc^2\sqrt{5c}$

d. $20bc^2\sqrt{ac}$

10. Solve $4x^2 + 5 = 41$.

a. $x = +3$

b. $x = \pm 3$

c. $x = -3$

d. $x = \pm 9$

Review

Hour 22

Square Roots: Four Operations and Radical Equations

CHAPTER SUMMARY

In this hour you will learn about …

- Multiplying, dividing, and simplifying square roots
- Adding, subtracting, and simplifying square roots
- Solving simple radical equations

In last hour's introduction to square roots, you used multiplication and division as you simplified the radical terms. When you learned about polynomials—how to combine like terms and use the FOIL method for multiplying binomials, along with other operations—you worked from your knowledge of basic algebraic properties. In a similar way, working with more advanced square root expressions is based on many of the same methods you used to simplify basic radicals. By the end of this hour, you will be able to solve equations that include square roots, which some people would call a "radical" goal. The first step is to learn how the basic operations affect square root expressions.

Multiplying, Dividing, and Simplifying Square Roots

The method you used last hour to simplify radicals helps you understand how to multiply square roots. Look again at the simplifying steps from this earlier example:

$$\sqrt{400} = \sqrt{4 \cdot 100} = \sqrt{4} \cdot \sqrt{100} = 2 \cdot 10 = 20$$

You factored 400 into two perfect squares, split them into two radicals, simplified each one, and then multiplied to get 20. This is really just a multiplication problem, and the part you need to focus on now follows.

JUST A MINUTE

Remember that the radicand is the number or variable inside the radical, or square root symbol.

The factors of $\sqrt{4}$ and $\sqrt{100}$ can be multiplied, and the result is $\sqrt{400}$. Multiplying two or more radical expressions simply means multiplying their radicands and putting the product inside the square root symbol. If there are numbers or variables in front of a radical, they act much like coefficients—they are multipliers. The following examples show how to multiply radicals.

$$\sqrt{10} \cdot \sqrt{3} = \sqrt{30}$$

The factors of 30 ($2 \cdot 15$, $3 \cdot 10$, and $5 \cdot 6$) are not perfect squares, so $\sqrt{30}$ is the final answer.

$$2\sqrt{5} \cdot 6\sqrt{20} = 12 \cdot \sqrt{100} = 12 \cdot 10 = 120$$

The numbers in front of the radicals are multiplied, and the radicands are also multiplied. Because $\sqrt{100}$ simplifies to become 10, with no radical, the final answer is the product of 12 times 10. The final result of multiplying radicals might or might not include a radical in the answer.

In some problems, it is more efficient to simplify the radicals before you multiply the radicands. The following example is worked out with each of these methods:

$$3\sqrt{12} \cdot 4\sqrt{28}$$

You can multiply 3 times 4 to find the number outside the radical and 12 times 28 to find the number inside it. You then simplify the large radicand, if possible, as shown here:

$$3\sqrt{12} \cdot 4\sqrt{28} = 12\sqrt{336}$$

JUST A MINUTE

Square root expressions are simplified when all the possible perfect square factors of the radicand have been changed into numbers in front of the radical and the final radicand has no perfect square factor.

There is probably a perfect square factor of 336, but it is rather difficult to find.

A more efficient alternative is to factor the original radicands before you multiply:

$$3\sqrt{12} \cdot 4\sqrt{28} = 12\sqrt{4 \cdot 3} \cdot 4\sqrt{4 \cdot 7} = 12 \cdot 2\sqrt{3} \cdot 4 \cdot 2\sqrt{7} = 192\sqrt{21}$$

This second method is usually the easier way to get the final answer because you are working with smaller numbers, but both methods eventually lead to the same final answer.

Now try to multiply the following radicals, simplifying each answer fully. Check your answers on page 327.

1. $\sqrt{7} \cdot \sqrt{10} =$
2. $5\sqrt{6} \cdot 2\sqrt{3} =$
3. $6\sqrt{12} \cdot 3\sqrt{8} =$

Radicals that include fractions can also be multiplied, and the process is similar to multiplying other radicals. However, their final answers need to be simplified into a specific form. The following examples show the solution steps for multiplying fractional radicands.

Multiply and simplify: $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{5}{12}}$

The two fractions can be brought together in a single radical and then multiplied, as shown here:

$$\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{5}{12}} = \sqrt{\frac{2}{3} \cdot \frac{5}{12}} = \sqrt{\frac{10}{36}} = \frac{\sqrt{10}}{6}$$

JUST A MINUTE

Fractions can be multiplied when they have different denominators, but they do need equal denominators when they are added or subtracted.

The factors of 10 ($2 \cdot 5$) are not perfect squares, so this radical is simplified.

Special rules affect radicals with fractions. The final answer cannot have a fraction under the square root sign, and there cannot be a radical in the denominator of a fraction. It is acceptable for only the numerator to be inside the radical. The following examples show how to work with a fractional answer within a radical so that it meets these special requirements.

Multiply and simplify: $\sqrt{\frac{5}{3}} \cdot \sqrt{\frac{7}{2}}$

This problem can be combined easily by simply multiplying the fractional parts under one radical, as shown:

$$\sqrt{\frac{5}{3}} \cdot \sqrt{\frac{7}{2}} = \sqrt{\frac{5}{3} \cdot \frac{7}{2}} = \sqrt{\frac{35}{6}}$$

Because the factors of 35 ($5 \cdot 7$) and the factors of 6 ($2 \cdot 3$) are not perfect squares, the radical is simplified. However, a fraction cannot be within the radical, so this answer still needs a little work.

The steps that follow show how to force the denominator 6 to become a perfect square and, therefore, move out of the radical:

$$\sqrt{\frac{35}{6}} = \sqrt{\frac{35 \cdot 6}{6 \cdot 6}} = \sqrt{\frac{210}{36}} = \frac{\sqrt{210}}{6}$$

JUST A MINUTE

Any number divided by itself equals one. If you multiply a fraction by *x*/*x*, where *x* is any nonzero number, the new fraction is still equal to the old one because you multiplied it by 1.

Multiplying both the numerator and the denominator by 6 is the same as multiplying the fraction by 1, and $6 \cdot 6 = 36$, which is a perfect square. The final answer is totally simplified now because only the numerator of the fraction is inside the radical.

Another example of simplifying radicals that include fractions follows, with an alternate method of multiplication.

Multiply and simplify: $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{15}{2}}$

This problem combines the method from the previous section with the need to leave no denominator inside the radical. Two solution steps are shown, with both as excellent options.

$$\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{15}{2}} = \sqrt{\frac{3}{5} \cdot \frac{15}{2}} = \sqrt{\frac{45}{10}} = \sqrt{\frac{9 \cdot 5}{2 \cdot 5}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

In this solution, the fractions were multiplied and then factored and simplified. The denominator $\sqrt{2}$ needed to be multiplied by $\sqrt{2}$ to move

the 2 out of the radical, so the entire fraction was multiplied by $\frac{\sqrt{2}}{\sqrt{2}}$, which is equal to 1.

FYI In each problem where the denominator contains a radical, the entire fraction must be multiplied by a new fraction equal to 1. You can always use the denominator term as the multiplier, as long as you multiply both the numerator and the denominator by it, effectively multiplying by a fraction that equals 1.

A second solution process can also be used on this problem, as follows:

$$\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{15}{2}} = \sqrt{\frac{3}{5} \cdot \frac{15}{2}} = \sqrt{\frac{3}{5} \cdot \frac{5 \cdot 3}{2}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

This is only slightly easier than the previous method, but it does simplify the fraction within the radical at an earlier point, keeping the numbers smaller and possibly easier to use, with fewer errors.

With either method, the end result is the same and requires moving the denominator outside the radical. As you see in the prior examples and in the next problem that follows, this complex algebraic process demands an understanding of square roots and fractions in the same problem, and it forces you to think of a logical process that can most efficiently meet your goal of simplifying the radical.

Multiply and simplify: $\sqrt{\frac{8}{3}} \cdot \sqrt{\frac{5}{4}}$

The 8 in the numerator and the 4 in the denominator have a common factor of 4, so they simplify easily, making this problem less complex. The denominator still needs to be moved out of the radical to be totally simplified:

$$\sqrt{\frac{8}{3}} \cdot \sqrt{\frac{5}{4}} = \sqrt{\frac{8}{3} \cdot \frac{5}{4}} = \sqrt{\frac{2 \cdot 4}{3} \cdot \frac{5}{4}} = \sqrt{\frac{10}{3}} = \sqrt{\frac{10}{3} \cdot \frac{3}{3}} = \frac{\sqrt{30}}{3}$$

Now try to multiply and simplify the following radicals, checking your answers on page 327.

4. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{7}{12}} =$

5. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{1}{2}} =$

6. $\sqrt{\frac{6}{7}} \cdot \sqrt{\frac{10}{3}} =$

Many radical problems include a combination of multiplication and addition or subtraction, so let's look at some basic adding and subtracting examples before putting them all together.

JUST A MINUTE

Recall that when adding or subtracting polynomials, only like terms can be combined: they must have the same variable and exponent. The coefficients are combined, with no change to the variable or exponent. Adding or subtracting radicals is very similar to polynomials, but instead of variables with exponents, you have square root expressions.

As you saw in the earlier problems, you don't need to have like terms to multiply radicals. In fact, you can put multiplied radicals together into one large square root symbol and simplify the problem from there.

When you are adding and subtracting radicals, there are a few more limitations, but your ability to simplify polynomials gives all the groundwork you need, as shown in the following examples:

Add and simplify: $3\sqrt{2} + 5\sqrt{2}$

Both terms in this addition problem have the same $\sqrt{2}$ radical. If this problem were $3x + 5x$, you would know the answer is $8x$. Similarly, you can just combine the coefficients in the problem, and it's done.

$$3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$$

As long as the radical is the same for the terms, the addition or subtraction can be simplified into a single term, as shown here:

$$6\sqrt{5} + 5\sqrt{5} - 2\sqrt{5} = 9\sqrt{5}$$

and

$$7\sqrt{3} + 2\sqrt{3} - \sqrt{3} = 8\sqrt{3}$$

In the previous example, the last term does not have a coefficient showing, but as with terms like x or n, these all have a coefficient of 1, which is seldom written.

FYI Notice that the radicands never change in addition or subtraction, but they do in multiplication and division. This is similar to how terms behave in polynomials, too.

When there are unlike radicals in an addition or subtraction problem, combine any that are the same and write the simplified answer with separate radicals, as in the next example:

$$5\sqrt{2} + 7\sqrt{6} - 8\sqrt{2} - 2\sqrt{3} =$$

The two like terms can be combined, but be careful—the radicands of 6, 2, and 3 are definitely not like terms.

$$5\sqrt{2} + 7\sqrt{6} - 8\sqrt{2} - 2\sqrt{3} = -3\sqrt{2} - 2\sqrt{3} + 7\sqrt{6}$$

Two of the terms were combined, and the answer was written with ascending radicands, but any order is acceptable, as long as each term's sign is correct. The nice part about adding and subtracting radicals is that their rules are similar to the rules for polynomials, and you already know them.

The final examples of adding and subtracting use your skills from simplifying radicals.

Consider the following addition problem:

$$\sqrt{2} + 3\sqrt{8} + 2\sqrt{3} - 2\sqrt{18} =$$

All these radicands are different, so it looks like this problem cannot be simplified. However, look more closely. The second and fourth radicals both have factors that are perfect squares, so they need to be simplified before any adding or subtracting can happen. The solution steps are shown in the following:

$$\begin{aligned}
&\sqrt{2} + 3\sqrt{8} + 2\sqrt{3} - 2\sqrt{18} = \\
&\sqrt{2} + 3\sqrt{4 \cdot 2} + 2\sqrt{3} - 2\sqrt{9 \cdot 2} = \\
&\sqrt{2} + 3 \cdot 2\sqrt{2} + 2\sqrt{3} - 2 \cdot 3\sqrt{2} = \\
&\sqrt{2} + 6\sqrt{2} + 2\sqrt{3} - 6\sqrt{2} = \\
&\sqrt{2} + 2\sqrt{3}
\end{aligned}$$

FYI Check each radicand for factors that are perfect squares so they can be moved out in front of the radical. This often gives new radicals that can be combined with others in the problem.

If you are able to take a few shortcuts, this process can be streamlined. Be aware that simplifying square roots can introduce mistakes, so show the work as needed. The following example includes larger radicands that need to be simplified:

$$2\sqrt{50} + 4\sqrt{32} - 2\sqrt{75} - 5\sqrt{90} =$$

Simplify each radical term and combine those that are like terms.

$$2\sqrt{50} + 4\sqrt{32} - 2\sqrt{75} - 5\sqrt{90} =$$
$$2\sqrt{25 \cdot 2} + 4\sqrt{16 \cdot 2} - 2\sqrt{25 \cdot 3} - 5\sqrt{9 \cdot 10} =$$
$$2 \cdot 5\sqrt{2} + 4 \cdot 4\sqrt{2} - 2 \cdot 5\sqrt{3} - 5 \cdot 3\sqrt{10} =$$
$$10\sqrt{2} + 16\sqrt{2} - 10\sqrt{3} - 15\sqrt{10} =$$
$$26\sqrt{2} - 10\sqrt{3} - 15\sqrt{10}$$

This last example is more difficult than most radical addition and subtraction problems, but even the complex radicals follow the same basic rules as the easier ones.

Now try to add and subtract the following radicals, checking your answers on page 328.

7. $2\sqrt{5} + 7\sqrt{3} - \sqrt{5} - 5\sqrt{3} =$
8. $\sqrt{20} + 4\sqrt{12} - 2\sqrt{27} =$

The next step is to combine your adding and subtracting skills with what you learned about multiplying radicals.

JUST A MINUTE

The distributive property makes it possible to simplify a sum multiplied by a number. It's the same as adding the product of the terms: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Consider the following sum of two radicals multiplied by another radical:

$$\sqrt{3}\left(\sqrt{3} + 5\right) =$$

The terms inside the parentheses cannot be combined because they are not like terms. However, the radical in front of the parentheses needs to be distributed, or multiplied, to both inner terms, as shown here:

$$\sqrt{3}\left(\sqrt{3} + 5\right) = \sqrt{3} \cdot \sqrt{3} + \sqrt{3} \cdot 5 = 3 + 5\sqrt{3}$$

This example includes a shortcut that applies any time a radical is multiplied times itself: the radicand becomes the simplified answer. The following example shows the different ways this can be written:

$$\sqrt{3} \cdot \sqrt{3} = \left(\sqrt{3}\right)^2 = \sqrt{9} = 3$$

A final distributive problem has a similar process but needs more simplifying of the radicals.

$$\sqrt{6}\left(\sqrt{2} - \sqrt{3}\right) =$$

As the outer number multiplies by each of the numbers inside the parentheses, the resulting radicals have perfect square factors and need to be simplified, as shown here:

$$\sqrt{6}\left(\sqrt{2} - \sqrt{3}\right) = \sqrt{12} - \sqrt{18} = \sqrt{4 \cdot 3} - \sqrt{9 \cdot 2} = 2\sqrt{3} - 3\sqrt{2}$$

PROCEED WITH CAUTION

These combined operation problems are perfect examples that need each step of the solution shown in detail. The changing radicals and their factors, along with the simplified radicals, are easily lost if you skip too many steps.

Now try to simplify the following radicals that have combined operations, checking your answers on page 328.

9. $\sqrt{2}\left(3 - \sqrt{2}\right) =$

10. $\sqrt{3}\left(\sqrt{15} - \sqrt{6}\right) =$

You have seen that radicals are affected by the basic operations—addition, subtraction, multiplication, and division—in much the same way polynomials are. When radicals appear in an equation, your goal is the same as in any equation—to isolate the variable. Your familiarity with radicals during the previous hour and this one makes the next step an easy one.

Solving Simple Radical Equations

The final goal this hour is a familiar one. So many of the topics in algebra keep coming back to solving equations, and this hour is no different. Many formulas in physics and geometry include radicals as terms. The ability to solve radical equations opens your mind to new analytical abilities, which, after all, is one of the main reasons to study algebra. Your skills with squaring terms and simplifying radicals merge here as you learn how to solve radical equations.

Let's begin with a basic equation:

$$\sqrt{x} = 6$$

To isolate this x variable, it needs to be moved outside of the square root symbol. In all the other equations you learned how to solve, the four basic properties helped you move terms from one side of the equation to the other, and eventually you could isolate the variable.

In this problem, none of the four operations will help. In the same way that you can add, subtract, multiply, or divide both sides of an equation by a number or variable, there's one more operation you can perform on both sides of this equation: you can square them both. And the solution steps follow:

$$\sqrt{x} = 6$$
$$\left(\sqrt{x}\right)^2 = 6^2$$
$$x = 36$$

FYI Squaring a radical removes the square root sign and moves the radicand outside the radical, so there is no radical remaining.

To check, put 36 in for x in the original equation:

$\sqrt{x} = 6 \rightarrow \sqrt{36} = 6 \rightarrow 6 = 6$; therefore, $x = 36$ is the solution.

The shortcut you saw in the last section is a huge help in this kind of equation. Squaring both sides of this basic equation makes finding the solution very easy.

Now consider another radical equation:

$$\sqrt{3x} = 9$$

Isolating the x will again be easy if you square both sides of the equation:

$$\sqrt{3x} = 9$$
$$\left(\sqrt{3x}\right)^2 = 9^2$$
$$3x = 81$$
$$\frac{3x}{3} = \frac{81}{3}$$
$$x = 27$$

To check, put 27 in for x in the original equation:

$$\sqrt{3x} = 9 \rightarrow \sqrt{3 \cdot 27} = 9 \rightarrow \sqrt{81} = 9 \rightarrow 9 = 9$$

Therefore, $x = 27$ is the solution.

Even rather complex radicands become easy to work with after the radical is squared. An equation that at first looks very difficult might actually be easy to solve, as in the next example:

$$\sqrt{2x - 3} = 7$$

PROCEED WITH CAUTION

Be sure to square both sides of the equation before you try to move the terms inside the radical to the other side; the radicand's terms cannot be moved until after the square root symbol is removed by the squaring process.

Square both sides of this equation to help isolate the variable:

$$\sqrt{2x-3}=7$$
$$\left(\sqrt{2x-3}\right)^2=7^2$$
$$2x-3=49$$
$$2x-3+3=49+3$$
$$2x=52$$
$$2x\div 2=52\div 2$$
$$x=26$$

To check, put 26 in for x in the original equation:

$$\sqrt{2x-3}=7 \rightarrow \sqrt{2\cdot 26-3}=7 \rightarrow \sqrt{52-3}=7 \rightarrow \sqrt{49}=7 \rightarrow 7=7$$

Therefore, $x = 26$ is the solution.

Consider the following equation:

$$\sqrt{5x-4}-3=6$$

Isolating this variable requires squaring both sides, but the –3 needs to be moved to the right side before you can square any of the terms:

$$\sqrt{5x-4}-3=6$$
$$\sqrt{5x-4}-3+3=6+3$$
$$\sqrt{5x-4}=9$$
$$\left(\sqrt{5x-4}\right)^2=9^2$$
$$5x-4=81$$
$$5x-4+4=81+4$$
$$5x=85$$
$$\frac{5x}{5}=\frac{85}{5}$$
$$x=17$$

PROCEED WITH CAUTION

Some of the addition and division property steps shown here can be done mentally to make the solution more compact, but always check the signs of the terms as you work and check that the correct operation is used for isolating the variable. Taking shortcuts is fine but may lead to careless errors.

To check, put 17 in for x in the original equation:

$$\sqrt{5x - 4} - 3 = 6 \rightarrow \sqrt{5 \cdot 17 - 4} - 3 = 6 \rightarrow \sqrt{85 - 4} - 3 = 6 \rightarrow$$
$$\sqrt{81} - 3 = 6 \rightarrow 9 - 3 = 6 \rightarrow 6 = 6 =$$

Therefore, $x = 17$ is the solution.

All the detailed steps are shown in the previous example, but you would probably be able to omit some of them. Solving radical equations is exactly like solving most of the equations so far, with the added step of squaring both sides of the equation to remove the square root symbol.

The following equation has a coefficient with the radical, and its solution steps show how to move the coefficient in a method very similar to solving the equations you learned about in Hour 5:

$$2\sqrt{2x - 6} + 4 = 8$$

Looking at the radical as an entire variable helps you see that you must move the 4 and the coefficient 2 before you can square the radical and then isolate the x.

$$2\sqrt{2x - 6} + 4 = 8$$
$$2\sqrt{2x - 6} + 4 - 4 = 8 - 4$$
$$2\sqrt{2x - 6} = 4$$
$$\frac{2\sqrt{2x - 6}}{2} = \frac{4}{2}$$
$$\sqrt{2x - 6} = 2$$
$$\left(\sqrt{2x - 6}\right)^2 = 2^2$$
$$2x - 6 = 4$$
$$2x - 6 + 6 = 4 + 6$$
$$2x = 10$$
$$\frac{2x}{2} = \frac{10}{2}$$
$$x = 5$$

To check, put 5 in for x in the original equation:

$$2\sqrt{2x - 6} + 4 = 8 \rightarrow 2\sqrt{2 \cdot 5 - 6} + 4 = 8 \rightarrow 2\sqrt{10 - 6} + 4 = 8 \rightarrow$$
$$2\sqrt{4} + 4 = 8 \rightarrow 2 \cdot 2 + 4 = 8 \rightarrow 8 = 8$$

Therefore, $x = 5$ is the solution.

GO TO ◄
Many of the common algebraic symbols and operations that represent word problem expressions are listed in Hour 1.

Square root problems can be solved when they are written as equations, but it can be difficult to know what terms are inside the radical, what parts are acting as coefficients, and what terms are added or subtracted outside the radical. The following word problems include different phrases and their algebraic translations.

If 4 times the square root of a number is 12, what is the number?

To write this as an equation, focus on what needs to be in the square root symbol. The key phrase is "the square root of a number" If you use x as the variable, x will be inside the radical, and the rest of the problem is easily translated:

$$4\sqrt{x} = 12$$
$$\frac{4\sqrt{x}}{4} = \frac{12}{4}$$
$$\sqrt{x} = 3$$
$$\left(\sqrt{x}\right)^2 = 3^2$$
$$x = 9$$

To check, put 9 in for x in the original equation:

$$4\sqrt{x} = 12 \rightarrow 4\sqrt{9} = 12 \rightarrow 4 \cdot 3 = 12 \rightarrow 12 = 12$$

Therefore, $x = 9$ is the solution.

Look for the key phrase in the following problem:

If the square root of 4 times a number is increased by 7 to give a result of 17, what is the number?

The key phrase in this equation is "the square root of 4 times a number." So $4 \cdot x$ will be inside the radical, as shown here:

$$\sqrt{4x} + 7 = 17$$
$$\sqrt{4x} + 7 - 7 = 17 - 7$$
$$\sqrt{4x} = 10$$
$$\left(\sqrt{4x}\right)^2 = 10^2$$
$$4x = 100$$
$$\frac{4x}{4} = \frac{1000}{4}$$
$$x = 25$$

To check, put 25 in for x in the original equation:

$$\sqrt{4x} + 7 = 17 \rightarrow \sqrt{4 \cdot 25} + 7 = 17 \rightarrow \sqrt{100} + 7 = 17 \rightarrow$$
$$10 + 7 = 17 \rightarrow 17 = 17$$

Therefore, $x = 25$ is the solution.

PROCEED WITH CAUTION

Pay close attention to the placement of the terms in a radical equation: "4 times the square root of a number" is very different from "The square root of 4 times a number," and each gives very different results.

As in all word problems, being able to write the equation involves knowing what algebraic symbols and operations match the word phrases. You began this process in Hour 1, and it has expanded to include everything from polynomial expressions to radicals.

Now try to solve the following radical equations, checking your answers on page 328.

11. $2\sqrt{x} = 6$
12. $\sqrt{x + 2} = 6$
13. $\sqrt{2x} = 6$

Answers to Sample Problems

1. $\sqrt{7} \cdot \sqrt{10} = \sqrt{7 \cdot 10} = \sqrt{70}$
2. $5\sqrt{6} \cdot 2\sqrt{3} = 10\sqrt{6 \cdot 3} = 10\sqrt{18} = 10\sqrt{9 \cdot 2} = 10 \cdot 3 \cdot \sqrt{2} = 30\sqrt{2}$
3. $6\sqrt{12} \cdot 3\sqrt{8} = 18 \cdot \sqrt{12} \cdot \sqrt{8} = 18 \cdot \sqrt{4 \cdot 3} \cdot \sqrt{4 \cdot 2} =$
   $18 \cdot \sqrt{4 \cdot 4 \cdot 3 \cdot 2} = 18 \cdot 4\sqrt{6} = 72\sqrt{6}$
4. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{7}{12}} = \sqrt{\frac{2}{3} \cdot \frac{7}{12}} = \sqrt{\frac{14}{36}} = \frac{\sqrt{14}}{6}$
5. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{1}{2}} = \sqrt{\frac{3}{5} \cdot \frac{1}{2}} = \sqrt{\frac{3}{10}} = \sqrt{\frac{3}{10} \cdot \frac{10}{10}} = \sqrt{\frac{30}{100}} = \frac{\sqrt{30}}{10}$
6. $\sqrt{\frac{6}{7}} \cdot \sqrt{\frac{10}{3}} = \sqrt{\frac{6}{7} \cdot \frac{10}{3}} = \sqrt{\frac{3 \cdot 2}{7} \cdot \frac{10}{3}} = \sqrt{\frac{1 \cdot 2}{7} \cdot \frac{10}{1}} = \sqrt{\frac{20}{7}} =$
   $\sqrt{\frac{4 \cdot 5}{7} \cdot \frac{7}{7}} = \frac{2\sqrt{35}}{7}$

7. $2\sqrt{5} + 7\sqrt{3} - \sqrt{5} - 5\sqrt{3} = \sqrt{5} + 2\sqrt{3}$
8. $\sqrt{20} + 4\sqrt{12} - 2\sqrt{27} = \sqrt{4 \cdot 5} + 4\sqrt{4 \cdot 3} - 2\sqrt{9 \cdot 3} = 2\sqrt{5} + 8\sqrt{3} - 6\sqrt{3} = 2\sqrt{5} + 2\sqrt{3}$
9. $\sqrt{2}(3 - \sqrt{2}) = 3\sqrt{2} - \sqrt{2} \cdot \sqrt{2} = 3\sqrt{2} - 2$
10. $\sqrt{3}(\sqrt{15} - \sqrt{6}) = \sqrt{3} \cdot \sqrt{15} - \sqrt{3} \cdot \sqrt{6} = \sqrt{45} - \sqrt{18} = \sqrt{9 \cdot 5} - \sqrt{9 \cdot 2} = 3\sqrt{5} - 3\sqrt{2}$
11. $2\sqrt{x} = 6 \rightarrow \frac{2\sqrt{x}}{2} = \frac{6}{2} \rightarrow \sqrt{x} = 3 \rightarrow (\sqrt{x})^2 = 3^2 \rightarrow x = 9$
12. $\sqrt{x+2} = 6 \rightarrow (\sqrt{x+2})^2 = 6^2 \rightarrow x + 2 = 36 \rightarrow x + 2 - 2 = 36 - 2 \rightarrow x = 34$
13. $\sqrt{2x} = 6 \rightarrow (\sqrt{2x})^2 = 6^2 \rightarrow 2x = 36 \rightarrow \frac{2x}{2} = \frac{36}{2} \rightarrow x = 18$

Review

Hour's Up!

Based on what you've learned in Hour 22, simplify the following square root expressions and solve the radical equations. Check your answers with the solution key in Appendix B.

1. $\sqrt{3} \cdot \sqrt{7} =$
- **a.** $3\sqrt{7}$
- **b.** $\sqrt{21}$
- **c.** $7\sqrt{3}$
- **d.** $\sqrt{10}$

2. $2\sqrt{2} \cdot 7\sqrt{6} =$
- **a.** $56\sqrt{3}$
- **b.** $9\sqrt{12}$
- **c.** $14\sqrt{6}$
- **d.** $28\sqrt{3}$

3. $3\sqrt{20} \cdot \sqrt{18} =$
- **a.** $3\sqrt{38}$
- **b.** $8\sqrt{10}$
- **c.** $18\sqrt{10}$
- **d.** $10\sqrt{10}$

Review

4. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{1}{20}} =$

a. $\frac{\sqrt{3}}{100}$

b. $\frac{\sqrt{4}}{10}$

c. $\frac{\sqrt{3}}{10}$

d. $\frac{\sqrt{3}}{5}$

5. $\sqrt{\frac{30}{7}} \cdot \sqrt{\frac{2}{5}} =$

a. $\frac{2\sqrt{42}}{3}$

b. $\frac{\sqrt{42}}{5}$

c. $\frac{\sqrt{32}}{\sqrt{12}}$

d. $\frac{2\sqrt{21}}{7}$

6. $5\sqrt{2} - 4\sqrt{3} - \sqrt{2} - 2\sqrt{3} =$

a. $5\sqrt{2} - 6\sqrt{3}$

b. $4\sqrt{2} - 6\sqrt{3}$

c. $4\sqrt{2} + 6\sqrt{3}$

d. $5\sqrt{2} + 6\sqrt{3}$

7. $2\sqrt{12} - 3\sqrt{20} + 2\sqrt{27} =$

a. $-\sqrt{8} + 2\sqrt{27}$

b. $10\sqrt{5} - 6\sqrt{3}$

c. $-2\sqrt{3} - 6\sqrt{5}$

d. $10\sqrt{3} - 6\sqrt{5}$

Review

8. $\sqrt{3}(5 - \sqrt{3}) =$

a. $5\sqrt{3} - 3$

b. $-4\sqrt{3}$

c. $\sqrt{15} - 3$

d. $5\sqrt{3}$

9. Solve: $3\sqrt{x} = 18$

a. $x = 6$

b. $x = 36$

c. $x = 18$

d. $x = 54$

10. Solve: $\sqrt{x - 3} = 7$

a. $x = 21$

b. $x = 49$

c. $x = 52$

d. $x = 46$

Hour 23

Quadratic Equations: Solving by Three Methods

Chapter Summary

In this hour you will learn about …

- Solving quadratic equations by finding square roots
- Solving quadratic equations by factoring
- Solving quadratic equations by completing the square

Knowing about square roots and radicals, how they are affected by the four basic operations, and how to solve radical equations prepares you for these final two hours of study. Solving an equation that has a squared variable relies on your ability to find the square root of a number and to simplify radical expressions. Factoring a polynomial will also come in handy.

To learn how to solve complex quadratic equations, you will be combining skills from many of the past 22 hours, especially the polynomial factoring in Hour 10 and the basic equations you solved in Hour 21. As with so many of the concepts in algebra, you will be expanding your mind's ability to see a problem, consider your options, and carry out a plan to find the solution. This has been the strategy all along: you began with basic math concepts and have progressed to sophisticated algebraic concepts, such as quadratic equations, which come in various forms and therefore have different solution options.

GO TO ◀
Recall that equations with a squared variable are called quadratic equations; see Hour 21 for the origins of the term quadratic.

Solving Quadratic Equations by Finding Square Roots

At the beginning of Hour 21, you learned about square roots, and by the end of that hour, you were able to solve equations like the following:

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm 5$$

and

$$3x^2 = 48$$
$$\frac{3x^2}{3} = \frac{48}{3}$$
$$x^2 = 16$$
$$\sqrt{x^2} = \sqrt{16}$$
$$x = \pm 4$$

JUST A MINUTE

Every quadratic equation has two solutions because taking the square root of a number gives both a positive and a negative result.

These two quadratic equations are perfect examples of why it's important to be able to take square roots and work with radicals. The second problem needs an extra step, but solving each equation involves taking the square root of both sides to isolate the variable x and then simplifying the radical to find the two solutions.

Now consider an equation that is quadratic, but the expression being squared isn't a simple x:

$$(x - 3)^2 = 25$$

The binomial $x + 3$ is squared, so this equation is very similar to $x^2 = 25$ and is solved in much the same way, as shown here:

$$(x - 3)^2 = 25$$
$$\sqrt{(x - 3)^2} = \sqrt{25}$$
$$x - 3 = \pm 5$$
$$x - 3 + 3 = \pm 5 + 3$$
$$x = 3 \pm 5$$

When you take the square root of 25, which is ±5, and then isolate the x by adding 3 to both sides of the equation, the solution is 3 ± 5. This needs to be simplified further to show the two solutions:

$$x = 3 \pm 5 \rightarrow x = 3 + 5 = 8 \text{ or } x = 3 - 5 = -2$$

To check, put 8 in the original equation for x:

$$(x - 3)^2 = 25 \rightarrow (8 - 3)^2 = 25 \rightarrow 5^2 = 25 \rightarrow 25 = 25$$

Put –2 in the original equation for x:

$(x - 3)^2 = 25 \rightarrow (-2 - 3)^2 = 25 \rightarrow (-5)^2 = 25 \rightarrow 25 = 25$; therefore, $x = 8$ and $x = -2$ are the solutions to $(x - 3)^2 = 25$.

PROCEED WITH CAUTION

If a binomial is squared in the equation, there will be two solutions, but do not assume that the final answers are opposites of each other.

The previous example includes a binomial—$x - 3$—that is squared. You learned how to multiply this expression with the FOIL process in Hour 8, but by leaving the binomial in its squared form, you can take the square root and quickly isolate the variable. FOILing the binomial would not be helpful in this problem.

Follow the solution steps for a more complex quadratic equation that includes a squared binomial term:

$$3(x + 2)^2 = 108$$

To begin this problem's solution process, the 3 must be divided on both sides, and then you can take the square root of both sides.

$$3(x + 2)^2 = 108$$

$$\frac{3(x + 2)^2}{3} = \frac{108}{3}$$

$$(x + 2)^2 = 36$$

$$\sqrt{(x + 2)^2} = \sqrt{36}$$

$$x + 2 = \pm 6$$

$$x + 2 - 2 = \pm 6 - 2$$

$$x = -2 \pm 6$$

The solution step can be written as $\pm 6 - 2$ or -2 ± 6, but the second form is easier to use when finding the two solutions, as follows:

$$x = -2 \pm 6 \rightarrow x = -2 + 6 = 4 \text{ or } x = -2 - 6 = -8$$

JUST A MINUTE

Remember that a number in front of a squared expression in parentheses is not squared; only the expression inside the parentheses is squared, and then the number in front is multiplied but not squared.

To check, put 4 in the original equation for x:

$$3(x + 2)^2 = 108 \rightarrow 3(4 + 2)^2 = 108 \rightarrow 3 \cdot (6)^2 = 108 \rightarrow 3 \cdot 36 = 108 \rightarrow 108 = 108$$

Put -8 in the original equation for x:

$$3(x + 2)^2 = 108 \rightarrow 3(-8 + 2)^2 = 108 \rightarrow 3 \cdot (-6)^2 = 108 \rightarrow 3 \cdot 36 = 108 \rightarrow 108 = 108$$; therefore, $x = 4$ and $x = -8$ are the solutions to $3(x + 2)^2 = 108$.

The following quadratic equation is solved with the same process as in the two earlier problems, with an additional step as you find the two solutions. It shows the importance of being able to simplify a square root that isn't a perfect square.

$$5(x - 4)^2 = 60$$

$$\frac{5(x - 4)^2}{5} = \frac{60}{5}$$

$$(x - 4)^2 = 12$$

$$\sqrt{(x - 4)^2} = \sqrt{12}$$

$$x - 4 = \pm\sqrt{12}$$

$$x - 4 + 4 = \pm\sqrt{12} + 4$$

$$x = 4 \pm \sqrt{12} = 4 \pm \sqrt{4 \cdot 3} = 4 \pm 2\sqrt{3}$$

The two solutions are $4 + 2\sqrt{3}$ and $4 - 2\sqrt{3}$.

PROCEED WITH CAUTION

When you square a radical, the result is the radicand—the number or expression inside the radical. If a number is in front of the radical and inside the parentheses, this number is squared and then multiplied times the expression that was under the radicand.

To check, put $4 + 2\sqrt{3}$ in the original equation for x:

$$5(x-4)^2 = 60 \rightarrow 5\left(4+2\sqrt{3}-4\right)^2 = 60 \rightarrow 5\cdot\left(2\sqrt{3}\right)^2 = 60 \rightarrow$$

$$5\cdot(4\cdot 3) = 60 \rightarrow 5\cdot 12 = 60 \rightarrow 60 = 60$$

Put $4 - 2\sqrt{3}$ in the original equation for x:

$$5(x-4)^2 = 60 \rightarrow 5\left(4-2\sqrt{3}-4\right)^2 = 60 \rightarrow 5\cdot\left(-2\sqrt{3}\right)^2 = 60 \rightarrow$$

$$5\cdot(4\cdot 3) = 60 \rightarrow 5\cdot 12 = 60 \rightarrow 60 = 60$$

Therefore, $x = 4 + 2\sqrt{3}$ and $x = 4 - 2\sqrt{3}$ are the solutions to $5(x - 4)^2 = 60$.

Although checking the solutions to this type of quadratic equation is almost as complex as solving the equation, the process for a difficult problem is no different from an easier one. The algebra skills required are more advanced, but the basic solution steps remain the same in all the equations and in their checks.

Now try to solve the following quadratic equations, checking your answers on page 346.

1. $(x - 8)^2 = 49$
2. $2(x + 4)^2 = 50$
3. $4(x + 2)^2 = 32$

The quadratic equations so far have included binomial terms that are squared. When the squared binomial is isolated, you can easily take the square root of both sides and use basic equation-solving skills to find the solutions.

Solving Quadratic Equations by Factoring

Consider the following quadratic equation that looks different from the others:

$$x^2 + 10x + 25 = 49$$

It does have a squared variable x, and the 25 could be subtracted from both sides, but the $10x$ term is a problem. If you subtract $10x$ from both sides, it will move to the right side of the equation, which stops you from taking the square root of both sides to isolate the x.

A better plan is to recall from your work with factoring polynomials that it might be possible to factor the trinomial $x^2 + 10x + 25$ into two binomials.

The last term in $x^2 + 10x + 25$ is 25, and its factors are $1 \cdot 25$ and $5 \cdot 5$. Does one of these pairs sum to 10 (the coefficient of the middle term)? Yes, $5 + 5 = 10$. From this work, you know how to factor $x^2 + 10x + 25$, as follows:

$$x^2 + 10x + 25 = (x + 5)(x + 5)$$

GO TO ◄

When factoring trinomials, focus first on the third term and find all the number pairs that multiply to give that term. Then choose the pair that sums to the coefficient of the trinomial's middle term. See Hour 10 for more examples of factoring trinomials.

Because $(x + 5)(x + 5) = (x + 5)^2$, this squared binomial can be used in the original equation and will help you get to the solution, as follows:

$$x^2 + 10x + 25 = 49$$
$$(x + 5)(x + 5) = 49$$
$$(x + 5)^2 = 49$$
$$\sqrt{(x + 5)^2} = \sqrt{49}$$
$$x + 5 = \pm 7$$
$$x + 5 - 5 = \pm 7 - 5$$
$$x = -5 \pm 7$$

By factoring the trinomial into a squared binomial and taking the square root of both sides of the equation, you can easily find the two solutions:

$$x = -5 \pm 7 \rightarrow x = -5 + 7 = 2 \text{ or } x = -5 - 7 = -12$$

To check, put 2 in the original equation for x:

$$x^2 + 10x + 25 = 49 \rightarrow 2^2 + 10 \cdot 2 + 25 = 49 \rightarrow 4 + 20 + 25 = 49 \rightarrow 49 = 49$$

Put –12 in the original equation for x:

$$x^2 + 10x + 25 = 49 \rightarrow (-12)^2 + 10 \cdot -12 + 25 = 49 \rightarrow 144 + -120 + 25 = 49 \rightarrow 49 = 49$$

; therefore, $x = 2$ and $x = -12$ are the solutions to $x^2 + 10x + 25 = 49$

JUST A MINUTE

As with all negative numbers that are squared, be sure you put parentheses around the number and its negative sign. This confirms that the answer will be a positive number, because multiplying two negatives always gives a positive result. $(-9) \cdot (-9) = +81$; $-9 \cdot 9 = -81$

The solution steps for the next quadratic equation are shown. If you can factor the trinomial into a squared binomial, you can find the solutions:

$$x^2 + 8x + 16 = 64$$

The last term in the trinomial is 16, and its factors are $1 \cdot 16$, $2 \cdot 8$, and $4 \cdot 4$. Does one of these pairs sum to 8? Yes, $4 + 4 = 8$, so the numbers in the binomials are 4 and 4, as shown in the following, along with the solution steps:

$$x^2 + 8x + 16 = 64$$
$$(x + 4)(x + 4) = 64$$
$$(x + 4)^2 = 64$$
$$\sqrt{(x + 4)^2} = \sqrt{64}$$
$$x + 4 = \pm 8$$
$$x + 4 - 4 = \pm 8 - 4$$
$$x = -4 \pm 8$$
$$x = -4 + 8 = 4 \quad \text{or} \quad x = -4 - 8 = -12$$

To check, put 4 in the original equation for x:

$$x^2 + 8x + 16 = 64 \rightarrow 4^2 + 8 \cdot 4 + 16 = 64 \rightarrow 16 + 32 + 16 = 64 \rightarrow 64 = 64$$

Put –12 in the original equation for x:

$$x^2 + 8x + 16 = 64 \rightarrow (-12)^2 + 8 \cdot -12 + 16 = 64 \rightarrow 144 - 96 + 16 = 64 \rightarrow 64 = 64$$

Therefore, $x = 4$ and $x = -12$ are the solutions to $x^2 + 8x + 16 = 64$.

The following quadratic equation has a trinomial with a negative middle term, but it can be factored into a squared binomial, as follows:

$$x^2 - 12x + 36 = 16$$

The factors of 36 are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$, and $6 \cdot 6$. Because $6 + 6 = 12$, the numbers in the binomials are 6 and 6, but they need to be –6 and –6 to sum to –12. The factoring and solution steps follow:

$$x^2 - 12x + 36 = 16$$
$$(x - 6)(x - 6) = 16$$
$$(x - 6)^2 = 16$$
$$\sqrt{(x - 6)^2} = \sqrt{16}$$
$$x - 6 = \pm 4$$
$$x - 6 + 6 = \pm 4 + 6$$
$$x = 6 \pm 4$$
$$x = 6 + 4 = 10 \quad \text{or} \quad x = 6 - 4 = 2$$

To check, put 10 in the original equation for *x:*

$$x^2 - 12x + 36 = 16 \rightarrow 2^2 - 12 \cdot 10 + 36 = 16 \rightarrow 100 - 120 + 36 = 16 \rightarrow 16 = 16$$

Put 2 in the original equation for *x:*

$$x^2 - 12x + 36 = 16 \rightarrow 2^2 - 12 \cdot 2 + 36 = 16 \rightarrow 4 - 24 + 36 = 16 \rightarrow 16 = 16$$

Therefore, $x = 10$ and $x = 2$ are the solutions to $x^2 - 12x + 36 = 16$.

JUST A MINUTE

Recall from Hour 2 that the algebraic term for the negative and positive whole numbers and zero is **integers.** Another way to describe integers is that they are zero and all the whole numbers along with their opposites. A number line often shows intervals that are marked in integers: …, –3, –2, –1, 0, +1, +2, +3, ….

The final example of a quadratic equation that can be solved by factoring is very similar to the earlier ones, but its solutions include a radical that cannot be simplified to a positive or negative integer or zero. Your simplifying radical skills are helpful in this problem.

$$x^2 - 6x + 9 = 20$$

The factors of 9 are $1 \cdot 9$ and $3 \cdot 3$. $-3 + -3 = -6$, so use -3 and -3 for the numbers in the binomials:

$$x^2 - 6x + 9 = 20$$
$$(x - 3)(x - 3) = 20$$
$$(x - 3)^2 = 20$$
$$\sqrt{(x - 3)^2} = \sqrt{20}$$
$$x - 3 = \pm\sqrt{4 \cdot 5}$$
$$x - 3 = \pm 2\sqrt{5}$$
$$x - 3 + 3 = \pm 2\sqrt{5} + 3$$
$$x = 3 \pm 2\sqrt{5}$$
$$x = 3 + 2\sqrt{5} \quad \text{or} \quad x = 3 - 2\sqrt{5}$$

The solution steps for this equation follow the same process as the other quadratic equations in this section, but checking the solutions requires skills that are more advanced and are not shown here. Your algebra skills of simplifying radicals do, however, allow you to isolate the variable and find the two solutions, which is your goal.

Now try to solve the following quadratic equations that include trinomials, checking your answers on page 346.

4. $x^2 - 10x + 25 = 81$
5. $x^2 - 4x + 4 = 12$

When a quadratic equation can be written with either a simple variable x that is squared or a binomial that is squared, taking the square root of both sides of the equation leads you easily to the two solutions. Let's look at your options if the trinomial in the equation cannot be factored into a squared binomial.

Solving Quadratic Equations by Completing the Square

Factoring the trinomial in a quadratic equation into two equal binomials makes it easy to find the two solutions.

However, for the following equation this doesn't seem to work:

$$x^2 + 6x + 4 = 11$$

The factors of 4 are $1 \cdot 4$ and $2 \cdot 2$. However, $1 + 4 = 5$ and $2 + 2 = 4$. Neither factor pair sums to 6, the middle term's coefficient, so neither of these factor pairs will work as the binomial factors for $x^2 + 6x + 4$.

When this happens—as it often does—you need to write the equation in such a way that the trinomial does factor into two equal binomials. You need to look at the original equation in a new way.

To understand the new process you'll be using, it is helpful to analyze the equations from the last section, when you learned how to solve quadratic equations by factoring.

Look at the trinomials from those equations and see how they were factored into squared binomials:

$$x^2 + 10x + 25 \rightarrow (x + 5)(x + 5) \rightarrow (x + 5)^2$$
$$x^2 + 8x + 16 \rightarrow (x + 4)(x + 4) \rightarrow (x + 4)^2$$
$$x^2 - 12x + 36 \rightarrow (x - 6)(x - 6) \rightarrow (x - 6)^2$$
$$x^2 - 10x + 25 \rightarrow (x - 5)(x - 5) \rightarrow (x - 5)^2$$
$$x^2 - 6x + 9 \rightarrow (x - 3)(x - 3) \rightarrow (x - 3)^2$$

In each problem, compare the middle term's coefficient to the last term. What can you do to 10 to get 25? What can you do to 8 to get 16? What can you do to –12 to get 36?

You might notice a pattern here: in each case, if you take half the middle terms' coefficient and then square that number, you get the last term. This is not a coincidence.

In fact, it is the key to solving an equation like the one that you are working on:

$$x^2 + 6x + 4 = 11$$

JUST A MINUTE

The addition property of equality or the subtraction property of equality helps you move terms that are added or subtracted to the other side of the equation. These properties help you move quadratic equation terms to make their trinomial expressions factorable.

To begin, move the 4 to the right side of the equation by subtracting 4 from both sides. Leave a blank space where the 4 was and include a +__ after the 7, as shown here:

$$x^2 + 6x + 4 = 11$$
$$x^2 + 6x + 4 - 4 = 11 - 4$$
$$x^2 + 6x + __ = 7 + __$$

Look only at the trinomial, ignoring the = 7 +__ for now:

$$x^2 + 6x + __$$

Focus on the middle term's coefficient, which is 6.

Take half of 6 → 3. Square 3 → 9.

In the trinomial $x^2 + 6x + __$, put 9 in the blank.

It is now $x^2 + 6x + 9$, and you can factor it:

$$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$$

When you added 9 to $x^2 + 6x$ and created a trinomial that factors into $(x + 3)(x + 3)$, you "completed the square." Moving the 4 gave you the space to add 9. What had been a trinomial that you couldn't factor—$x^2 + 6x + 4$—became a trinomial that factored into a squared binomial—$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$.

Completing the square is a technique that puts you in control of a quadratic equation that otherwise you can't solve by factoring. Let's go back to the equation and solve it. The solution steps that include completing the square follow:

$$x^2 + 6x + 4 = 11$$
$$x^2 + 6x + 4 - 4 = 11 - 4$$
$$x^2 + 6x + __ = 7 + __$$
$$x^2 + 6x + 9 = 7 + 9$$

PROCEED WITH CAUTION

The most common mistake is not adding the number to both sides of the equation. It is easy to add the number that is needed to make the trinomial factorable into a squared binomial and also easy to forget to add that same number to the right side. Using the +___ on the right side of the equation helps remind you to add the number to both sides.

When you add 9 to complete the trinomial, you must also add 9 to the other side of the equation, an example of the addition property of equality you've been using since Hour 4. To finish the solution steps:

$$x^2 + 6x + 9 = 7 + 9$$
$$(x + 3)(x + 3) = 16$$
$$(x + 3)^2 = 16$$
$$\sqrt{(x + 3)^2} = \sqrt{16}$$
$$x + 3 = \pm 4$$
$$x + 3 - 3 = \pm 4 - 3$$
$$x = -3 \pm 4$$
$$x = -3 + 4 = 1 \quad \text{or} \quad x = -3 - 4 = -7$$

To check, put 1 in the original equation for x:

$$x^2 + 6x + 4 = 11 \rightarrow 1^2 + 6 \cdot 1 + 4 = 11 \rightarrow 1 + 6 + 4 = 11 \rightarrow 11 = 11$$

Put –7 in the original equation for x:

$$x^2 + 6x + 4 = 11 \rightarrow (-7)^2 + 6 \cdot -7 + 4 = 11 \rightarrow 49 - 42 + 4 = 11 \rightarrow 11 = 11$$

Therefore, $x = 1$ and $x = -7$ are the solutions to $x^2 + 6x + 4 = 11$.

Follow the solution steps for another example of completing the square to solve a quadratic equation:

$$x^2 - 8x + 3 = 12$$
$$x^2 - 8x + 3 - 3 = 12 - 3$$
$$x^2 - 8x + __ = 9 + __$$
$$x^2 - 8x + 16 = 9 + 16$$
$$(x - 4)(x - 4) = 25$$
$$(x - 4)^2 = 25$$
$$\sqrt{(x - 4)^2} = \sqrt{25}$$
$$x - 4 = \pm 5$$
$$x - 4 + 4 = \pm 5 + 4$$
$$x = 4 \pm 5$$
$$x = 4 + 5 = 9 \quad \text{or} \quad x = 4 - 5 = -1$$

FYI If the middle term's coefficient is negative, when you take half a negative number, the result is still negative. However, squaring that negative number will result in a positive number. In all equations solved by completing the square, you will always be adding a term to both sides of the equation, regardless of the sign of the middle term's coefficient.

To check, put 9 in the original equation for x:

$$x^2 - 8x + 3 = 12 \rightarrow (9)^2 - 8 \cdot 9 + 3 = 12 \rightarrow 81 - 72 + 3 = 12 \rightarrow 12 = 12$$

Put –1 in the original equation for x:

$$x^2 - 8x + 3 = 12 \rightarrow (-1)^2 - 8 \cdot -1 + 3 = 12 \rightarrow 1 + 8 + 3 = 12 \rightarrow 12 = 12$$

Therefore, $x = 9$ and $x = -1$ are the solutions to $x^2 - 8x + 3 = 12$.

What happens if the coefficient of the middle terms is not an even number? All the examples so far have been even. You still take half of the middle term's coefficient and square it to find the term that is added to both sides of the equation. However, taking half of an odd number does give a fraction, as you see in the following equation and its solution steps:

$x^2 + 7x - 8 = 10$
$x^2 + 7x - 8 + 8 = 10 + 8$
$x^2 + 7x + __ = 18 + __$
Take half of $7 \rightarrow \frac{7}{2}$.
Square $\frac{7}{2} \rightarrow \frac{49}{4}$.
Add $\frac{49}{4}$ to both sides of the equation to complete the square.

The remaining solution steps are shown here:

$$x^2 + 7x + \frac{49}{4} = 18 + \frac{49}{4}$$

$$\left(x + \frac{7}{2}\right)\left(x + \frac{7}{2}\right) = \frac{18 \cdot 4}{4} + \frac{49}{4}$$

$$\left(x + \frac{7}{2}\right)^2 = \frac{72}{4} + \frac{49}{4}$$

$$\left(x + \frac{7}{2}\right)^2 = \frac{121}{4}$$

$$\sqrt{\left(x + \frac{7}{2}\right)^2} = \sqrt{\frac{121}{4}}$$

$$x + \frac{7}{2} = \pm\frac{11}{2}$$

$$x + \frac{7}{2} - \frac{7}{2} = \pm\frac{11}{2} - \frac{7}{2}$$

$$x = -\frac{7}{2} + \frac{111}{2} = \frac{4}{2} = 2 \quad \text{or} \quad x = -\frac{7}{2} - \frac{11}{2} = -\frac{18}{2} = -9$$

To check, put 2 in the original equation for x:

$$x^2 + 7x - 8 = 10 \rightarrow (2)^2 + 7 \cdot 2 - 8 = 10 \rightarrow 4 + 14 - 8 = 10 \rightarrow 10 = 10$$

Put –9 in the original equation for x:

$$x^2 + 7x - 8 = 10 \rightarrow (-9)^2 + 7 \cdot -9 - 8 = 10 \rightarrow 81 - 63 - 8 = 10 \rightarrow 10 = 10$$

Therefore, $x = 2$ and $x = -9$ are the solutions to $x^2 + 7x - 8 = 10$.

As with the problems in the earlier section that were solved with factoring, some solutions will not be integers when you complete the square. The next example shows how to simplify the radical solutions of a quadratic equation solved by completing the square:

$$x^2 + 12x - 10 = 4$$
$$x^2 + 12x - 10 + 10 = 4 + 10$$
$$x^2 + 12x + __ = 14 + __$$
$$x^2 + 12x + 36 = 14 + 36$$
$$(x + 6)(x + 6) = 50$$
$$(x + 6)^2 = 50$$
$$\sqrt{(x + 6)^2} = \sqrt{50}$$
$$x + 6 = \pm\sqrt{50}$$
$$x + 6 = \pm\sqrt{25 \cdot 2}$$
$$x + 6 - 6 = \pm 5\sqrt{2} - 6$$
$$x = -6 \pm 5\sqrt{2}$$
$$x = -6 + 5\sqrt{2} \quad \text{or} \quad x = -6 - 5\sqrt{2}$$

Checking these solutions requires skills that are more advanced and not shown here.

Now try to solve the following quadratic equations by completing the square. Check your answers on pages 346–347.

6. $x^2 + 10x - 6 = 5$
7. $x^2 - 6x + 2 = 42$
8. $x^2 + 5x - 9 = 5$

Your ability to solve quadratic equations enables you to break difficult problems into manageable steps that lead to solutions. You can apply this skill to many different areas in life; looking at information from a new perspective can help you resolve many complex problems in a creative way.

Answers to Sample Problems

1. $(x-8)^2 = 49 \rightarrow \sqrt{(x-8)^2} = \sqrt{49} \rightarrow x - 8 = \pm 7 \rightarrow$ $x - 8 + 8 = \pm 7 + 8 \rightarrow x = 8 \pm 7 \rightarrow x = 8 + 7 = 15$ or $x = 8 - 7 = 1$

2. $2(x+4)^2 = 50 \rightarrow \frac{2(x+4)^2}{2} = \frac{50}{2} \rightarrow (x+4)^2 = 25 \rightarrow \sqrt{(x+4)^2}$ $= \sqrt{25} \rightarrow x + 4 = \pm 5 \rightarrow x + 4 - 4 = \pm 5 - 4 \rightarrow x = -4 \pm 5 \rightarrow x = -4$ $+ 5 = 1$ or $x = -4 - 5 = -9$

3. $4(x+2)^2 = 32 \rightarrow \frac{4(x+2)^2}{4} = \frac{32}{4} \rightarrow (x+2)^2 = 8 \rightarrow$ $\sqrt{(x+2)^2} = \sqrt{8} \rightarrow x + 2 = \pm\sqrt{8} \rightarrow x + 2 - 2 = \pm\sqrt{8} - 2 \rightarrow$ $x = -2 \pm \sqrt{8} \rightarrow x = -2 \pm \sqrt{4 \cdot 2} \rightarrow x = -2 \pm 2\sqrt{2} \rightarrow x =$ $-2 + 2\sqrt{2}$ or $x = -2 - 2\sqrt{2}$

4. $x^2 - 10x + 25 = 81 \rightarrow (x-5)(x-5) = 81 \rightarrow (x-5)^2 = 81 \rightarrow$ $\sqrt{(x-5)^2} = \sqrt{81} \rightarrow x - 5 = \pm 9 \rightarrow x - 5 + 5 = \pm 9 + 5 \rightarrow$ $x = 5 \pm 9 \rightarrow x = 5 + 9 = 14$ or $x = 5 - 9 = -4$

5. $x^2 - 4x + 4 = 12 \rightarrow (x-2)(x-2) = 12 \rightarrow (x-2)^2 = 12 \rightarrow$ $\sqrt{(x-2)^2} = \sqrt{12} \rightarrow x - 2 = \pm\sqrt{12} \rightarrow x - 2 + 2 = \pm\sqrt{12} + 2 \rightarrow$ $x = 2 \pm \sqrt{12} \rightarrow x = 2 \pm \sqrt{4 \cdot 3} \rightarrow x = 2 \pm 2\sqrt{3} \rightarrow x =$ $2 + 2\sqrt{3}$ or $x = 2 - 2\sqrt{3}$

6. $x^2 + 10x - 6 = 5 \rightarrow x^2 + 10x - 6 + 6 = 5 + 6 \rightarrow x^2 + 10x +$ $__ = 11 + __ \rightarrow x^2 + 10x + 25 = 11 + 25 \rightarrow (x+5)^2 = 36 \rightarrow$ $\sqrt{(x+5)^2} = \sqrt{36} \rightarrow x + 5 = \pm 6 \rightarrow x + 5 - 5 = \pm 6 - 5 \rightarrow x =$ $-5 \pm 6 \rightarrow x = -5 + 6 = 1$ or $x = -5 - 6 = -11$

7. $x^2 - 6x + 2 = 42 \rightarrow x^2 - 6x + 2 - 2 = 42 - 2 \rightarrow x^2 - 6x + __$ $= 40 + __ \rightarrow x^2 - 6x + 9 = 40 + 9 \rightarrow (x-3)(x-3) = 49 \rightarrow$ $(x-3)^2 = 49 \rightarrow \sqrt{(x-3)^2} = \sqrt{49} \rightarrow x - 3 = \pm 7 \rightarrow x - 3 + 3 =$ $\pm 7 + 3 \rightarrow x = 3 \pm 7 \rightarrow x = 3 + 7 = 10$ or $x = 3 - 7 = -4$

8. $x^2 + 5x - 9 = 5 \rightarrow x^2 + 5x - 9 + 9 = 5 + 9 \rightarrow x^2 + 5x + __ =$
$14 + __ \rightarrow x^2 + 5x + \frac{25}{4} = 14 + \frac{25}{4} \rightarrow \left(x + \frac{5}{2}\right)\left(x + \frac{5}{2}\right) =$
$\frac{14 \cdot 4}{4} + \frac{25}{4} \rightarrow \left(x + \frac{5}{2}\right)^2 = \frac{56 + 25}{4} \rightarrow \left(x + \frac{5}{2}\right)^2 = \frac{81}{4} \rightarrow$
$\sqrt{\left(x + \frac{5}{2}\right)^2} = \sqrt{\frac{81}{4}} \rightarrow x + \frac{5}{2} = \pm\frac{9}{2} \rightarrow x + \frac{5}{2} - \frac{5}{2} = \pm\frac{9}{2} - \frac{5}{2} \rightarrow$
$x = -\frac{5}{2} \pm \frac{9}{2} \rightarrow x = -\frac{5}{2} + \frac{9}{2} = \frac{-5 + 9}{2} = \frac{4}{2} = 2$ or
$x = -\frac{5}{2} - \frac{9}{2} = \frac{-5 - 9}{2} = \frac{-14}{2} = -7$

Hour's Up!

Review

Based on what you've learned in Hour 23, solve the following quadratic equations. Check your answers with the solution key in Appendix B.

1. $(x - 6)^2 = 25$
a. $x = -11$ or 1
b. $x = -11$ or -1
c. $x = 11$ or 1
d. $x = 11$ or -1

2. $3(x - 8)^2 = 48$
a. $x = -12$ or 4
b. $x = 12$ or 4
c. $x = 12$ or -4
d. $x = -12$ or -4

3. $4(x + 3)^2 = 24$
a. $x = 3 + \sqrt{6}$ or $3 - \sqrt{6}$
b. $x = 3 + \sqrt{24}$ or $3 - \sqrt{24}$
c. $x = -3 + \sqrt{6}$ or $-3 - \sqrt{6}$
d. $x = -3 + \sqrt{24}$ or $-3 - \sqrt{24}$

4. $x^2 + 12x + 36 = 49$
- **a.** $x = 1$ or -13
- **b.** $x = -1$ or 13
- **c.** $x = -1$ or -13
- **d.** $x = 1$ or 13

5. $x^2 - 8x + 16 = 36$
- **a.** $x = -10$ or -2
- **b.** $x = 10$ or -2
- **c.** $x = 10$ or 2
- **d.** $x = -10$ or 2

6. $x^2 + 10x + 25 = 16$
- **a.** $x = 9$ or -1
- **b.** $x = 9$ or 1
- **c.** $x = -9$ or -1
- **d.** $x = -9$ or 1

7. $x^2 - 4x + 4 = 100$
- **a.** $x = 12$ or 8
- **b.** $x = -12$ or 8
- **c.** $x = -12$ or -8
- **d.** $x = 12$ or -8

8. $x^2 + 8x + 12 = 32$
- **a.** $x = 2$ or -10
- **b.** $x = -2$ or -10
- **c.** $x = -2$ or 10
- **d.** $x = 2$ or 10

9. $x^2 + 10x + 20 = 95$
- **a.** $x = -5$ or 15
- **b.** $x = -5$ or -15
- **c.** $x = 5$ or 15
- **d.** $x = 5$ or -15

10. $x^2 - 6x + 3 = 75$
- **a.** $x = 12$ or 6
- **b.** $x = 12$ or -6
- **c.** $x = -12$ or -6
- **d.** $x = -12$ or 6

Review

Hour 24

Quadratic Equations: Solving by the Quadratic Formula

CHAPTER SUMMARY

In this hour you will learn about …

- Solving quadratic equations with the quadratic formula
- Choosing the best method for solving quadratic equations
- Finding the number and type of quadratic equation solutions
- Solving practical problems with quadratic equations

It's appropriate that your last hour of algebra study takes a closer look at quadratic equations. Equations are the heart and soul of algebra, and solving different types of quadratic equations brings together almost every topic you've studied during the past 23 hours. You'll see one more way to solve these equations, get some hints on which method is best to use, and work out practical problems that need quadratic equations to reach a solution.

The methods of solving quadratic equations that you learned last hour—finding the square root, factoring, and completing the square—will usually lead to the solutions. Some quadratic equations just don't work well with any of these three processes, so let's look at one more method.

Solving Quadratic Equations with the Quadratic Formula

One of the more difficult topics in algebra is solving a quadratic equation by completing the square of the trinomial expression. You learned this method last hour and had several opportunities to practice using it on equations that could not be solved by finding the square root or by factoring. You probably noticed, however, that the goal of completing the square is to force the problem to be a trinomial that factors into a squared binomial. Depending on the terms in the equation, this process of completing the square is either (a) difficult or (b) very difficult.

Fortunately, finding the square root or factoring will easily solve many quadratic equations, as in the following examples:

$$(x-7)^2 = 25$$
$$\sqrt{(x-7)^2} = \sqrt{25}$$
$$x - 7 = \pm 5$$
$$x - 7 + 7 = \pm 5 + 7$$
$$x = 7 \pm 5$$
$$x = 7 + 5 = 12 \quad \text{or} \quad x = 7 - 5 = 2$$

and

$$x^2 - 8x + 16 = 36$$
$$(x-4)^2 = 36$$
$$\sqrt{(x-4)^2} = \sqrt{36}$$
$$x - 4 = \pm 6$$
$$x - 4 + 4 = \pm 6 + 4$$
$$x = 4 \pm 6$$
$$x = 4 + 6 = 10 \quad \text{or} \quad x = 4 - 6 = -2$$

JUST A MINUTE

Any quadratic equation that has a single squared variable, a squared binomial, or a trinomial that factors into a squared binomial is easily solved by finding the square roots of both sides of the equation.

Completing the square of a trinomial that doesn't factor nicely into squared binomials can be used to find the equation's solutions. An alternative method—the quadratic formula—is based on completing the square and works efficiently on quadratic equations that can't be solved by either of the first two methods. Even better, it is much easier to use than completing the square. Let's look at a few basic facts about quadratic equations to see how this formula works.

Every quadratic equation has a squared variable, typically x. All quadratic equations with a variable x can be written in the following standard form:

$$ax^2 + bx + c = 0$$

In this general form, the a coefficient of x^2 cannot equal 0 because if $a = 0$, then ax^2 would also be 0, and the equation would not be quadratic; b and c can be any real number.

GO TO ◄
All quadratic equations have a squared variable and two solutions. See Hour 21 for details about basic quadratic equations.

If you begin with $ax^2 + bx + c = 0$ and follow a lengthy—9 or 10 steps—process of completing the square to solve for x, the result is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \text{the quadratic formula}$$

If you can write a quadratic equation in the general form of $ax^2 + bx + c = 0$, you can find the solution for x by putting the a, b, and c values from your equation into the previous quadratic formula. The ± sign in the numerator shows that there are two solutions, which you know to be true for all quadratic equations.

Consider the following quadratic equation:

$$x^2 + 6x + 4 = 11$$

As you try to solve this equation, you should consider all your options. You cannot take the square root of both sides (the $6x + 4$ is in the way, and moving them to the right side won't help), and the trinomial doesn't factor into a squared binomial (the factors of $4 = 1 \cdot 4$ and $4 = 2 \cdot 2$ don't sum to the middle term's coefficient of 6), so completing the square is your third option. But if you can make the original equation look like $ax^2 + bx + c = 0$, you can use the quadratic formula.

Can the equation $x^2 + 6x + 4 = 11$ be written so that it is in standard form, with all the terms on the left side of the equals sign and 0 on the right? If you subtract 11 from both sides, you will have 0 on the right side:

$$x^2 + 6x + 4 = 11$$
$$x^2 + 6x + 4 - 11 = 11 - 11$$
$$x^2 + 6x - 7 = 0$$

This last equation is in standard form, so the quadratic formula can be used to find the solutions. It helps to write your equation below the standard equation so you can see the a, b, and c values more clearly:

$$ax^2 + bx + c = 0$$
$$x^2 + 6x - 7 = 0$$

In this equation, $a = 1$, $b = 6$, and $c = -7$. Put them into the quadratic formula (notice that the sign of the b value changes as it is put into the formula) and simplify the expression, as shown in the following:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot -7}}{2 \cdot 1}$$

$$x = \frac{-6 \pm \sqrt{36 + 28}}{2}$$

$$x = \frac{-6 \pm \sqrt{64}}{2}$$

$$x = \frac{-6 \pm 8}{2}$$

$$x = \frac{-6 + 8}{2} = \frac{2}{2} = 1 \quad \text{or} \quad \frac{-6 - 8}{2} = \frac{-14}{2} = -7$$

JUST A MINUTE

Follow the order of operations you learned in Hour 1, working first to combine the terms inside the radical in the numerator.

To check, put 1 in for x in the original equation:

$$x^2 + 6x + 4 = 11 \rightarrow 1^2 + 6 \cdot 1 + 4 = 11 \rightarrow 1 + 6 + 4 = 11 \rightarrow 11 = 11$$

Put –7 in for x in the original equation:

$$x^2 + 6x + 4 = 11 \rightarrow (-7)^2 + 6 \cdot -7 + 4 = 11 \rightarrow 49 - 42 + 4 = 11 \rightarrow 11 = 11$$

Therefore, $x = 1$ and $x = -7$ are solutions of $x^2 + 6x + 4 = 11$.

FYI The quadratic formula looks difficult, but it is much easier to use than completing the square to solve a quadratic equation. Putting the ***a, b,*** and ***c*** values from your equation into the quadratic formula gives the two solutions.

The quadratic formula looks a little intimidating, but using it is often simpler than trying to complete the square of a trinomial. With completing the square, you must figure out what numbers will make the trinomial factorable as squared binomials, and if the middle term's coefficient is an odd number, you are very likely to make an error somewhere in the process.

The next quadratic equation is a perfect example of a problem most easily solved with the quadratic formula. Move $x + 30$ to the left side of the equals sign, and find the a, b, and c values. The solution steps are shown in the following:

$$x^2 = x + 30$$
$$x^2 - x - 30 = x + 3 - x - 30$$
$$x^2 - x - 30 = 0$$

Find the a, b, and c values

$$ax^2 + bx + c = 0$$
$$x^2 - x - 30 = 0$$

In this equation, $a = 1$, $b = -1$, and $c = -30$. Put them into the quadratic formula, watching the signs carefully, and then simplify the expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot -30}}{2 \cdot 1}$$

$$x = \frac{1 \pm \sqrt{1 + 120}}{2}$$

$$x = \frac{1 \pm \sqrt{121}}{2}$$

$$x = \frac{1 \pm 11}{2}$$

$$x = \frac{1 + 11}{2} = \frac{12}{2} = 6 \quad \text{or} \quad x = \frac{1 - 11}{2} = \frac{-10}{2} = -5$$

PROCEED WITH CAUTION

When the ***b*** value of your equation is negative, it will become positive in the numerator of the quadratic formula. Be careful to put parentheses around the negative ***b*** value when it is squared inside the radical. Squaring a negative number always gives a positive answer.

To check, put 6 in for x in the original equation:

$$x^2 = x + 30 \rightarrow 6^2 = 6 + 30 \rightarrow 36 = 36$$

Put –5 in for x in the original equation:

$$x^2 = x + 30 \rightarrow (-5)^2 = -5 + 30 \rightarrow 25 = 25$$

Therefore, $x = 6$ and $x = -5$ are solutions of $x^2 = x + 30$.

The solutions to the example equations so far have all been integers, but this is not always the case. As you use the quadratic formula to solve equations, simplifying the radical part of the numerator may result in a radical in the solutions. The following example shows this type of solution:

$$x^2 - 3x = 1$$
$$x^2 - 3x - 1 = 1 - 1$$
$$x^2 - 3x - 1 = 0$$

Find the a, b, and c values:

$$ax^2 + bx + c = 0$$
$$x^2 - 3x - 1 = 0$$

In this equation, $a = 1$, $b = -3$, and $c = -1$. Put them into the quadratic formula, watching the signs, and simplify the expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot -1}}{2 \cdot 1}$$

$$x = \frac{3 \pm \sqrt{9 + 4}}{2}$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$x = \frac{3 + \sqrt{13}}{2} \quad \text{or} \quad x = \frac{3 - \sqrt{13}}{2}$$

Checking these solutions requires skills that are more advanced, and the checking process is not shown here.

A final example of an equation solved with the quadratic formula includes a coefficient of 3 for the squared term. Its solution steps are shown here:

$$3x^2 = 5x + 2$$
$$3x^2 - 5x - 2 = 5x + 2 - 5x - 2$$
$$3x^2 - 5x - 2 = 0$$

Find the a, b, and c values:

$$ax^2 + bx + c = 0$$
$$3x^2 - 5x - 2 = 0$$

In this equation, $a = 3$, $b = -5$, and $c = -2$. Put them into the quadratic formula, watching the signs, and simplify the expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot -2}}{2 \cdot 3}$$

$$x = \frac{5 \pm \sqrt{25 + 24}}{6}$$

$$x = \frac{5 \pm \sqrt{49}}{6}$$

$$x = \frac{5 \pm 7}{6}$$

$$x = \frac{5 + 7}{6} = \frac{12}{6} = 2 \quad \text{or} \quad x = \frac{5 - 7}{6} = \frac{-2}{6} = -\frac{1}{3}$$

To check, put 2 in for x in the original equation:

$$3x^2 = 5x + 2 \rightarrow 3 \cdot 2^2 = 5 \cdot 2 + 2 \rightarrow 3 \cdot 4 = 10 + 2 \rightarrow 12 = 12$$

Put $-\frac{1}{3}$ in for x in the original equation:

$$3x^2 = 5x + 2 \rightarrow 3 \cdot \left(-\frac{1}{3}\right)^2 = 5 \cdot \left(-\frac{1}{3}\right) + 2 \rightarrow$$

$$3 \cdot \frac{1}{9} = -\frac{5}{3} + 2 \rightarrow \frac{3}{9} = -\frac{5}{3} + \frac{2 \cdot 3}{3} \rightarrow \frac{1}{3} = -\frac{5}{3} + \frac{6}{3} \rightarrow \frac{1}{3} = \frac{1}{3}$$

Therefore, $x = 2$ and $x = -\frac{1}{3}$ are solutions of $3x^2 = 5x + 2$.

Now try to solve the following quadratic equations using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Check your answers on page 365.

1. $x^2 + 2x = 15$
2. $6x^2 = -5x + 4$

The quadratic formula can be used to solve quadratic equations that don't fit the other methods. It can also be the method of choice for solving almost all quadratic equations. Knowing which of the four methods is the most efficient requires some analysis of the equation, but you'll see that some problems are clearly a certain type and that, with others, it will be your preference.

Choosing the Best Method for Solving Quadratic Equations

With four different ways to solve quadratic equations, how do you know which one to use? Look at how the following equations have certain characteristics. The method you use is based on the terms in the equation.

Each method is listed, followed by a few examples of equations that are easily solved by that method.

Method 1: Finding Square Roots

$$x^2 = 49$$
$$x^2 + 5 = 86$$
$$x^2 - 8 = 44$$
$$5x^2 = 125$$
$$\frac{x^2}{3} = 12$$

Each of the previous equations has a single x variable that is squared. After isolating the squared variable, take the square root of both sides of the equation to find the two solutions.

$$(x - 5)^2 = 36$$
$$3(x - 2)^2 = 48$$
$$5(x + 3)^2 - 7 = 13$$
$$2(x - 4)^2 + 8 = 58$$
$$\frac{()x + 4^{\ 2}}{2} = 8$$

Each of the previous equations has a variable within a squared binomial. After isolating the squared binomial, take the square root of both sides of the equation. Isolate the variable to find the two solutions.

Method 2: Factoring a Trinomial into Squared Binomials

$x^2 + 8x + 16 = 64$
$x^2 - 4x + 4 = 25$

Each of the previous equations has a trinomial that can be factored into a squared binomial. Taking the square root of both sides of the factored equation and isolating the variable give the two solutions.

Method 3: Completing the Square

$x^2 + 8x + 10 = 30$
$x^2 - 6x - 7 = 9$

The trinomial in each of the previous equations cannot be factored into a squared binomial. Isolating the x-squared and the x terms, adding the square of half the x term's coefficient to both sides, and factoring the new trinomial into a squared binomial gives an equation that can be solved with Method 2.

Method 4: The Quadratic Formula

$x^2 + 6x + 2 = 42$
$x^2 - 5x + 2 = -4$
$6x^2 + 7x + 10 = 13$

The trinomials in these three equations cannot be factored. The first two could be solved by Method 3, completing the square; however, the second equation's –5 coefficient on the x term leads to a fraction. The coefficient of 6 on the third equation's squared term makes it a perfect candidate for the quadratic formula solution method. Unless you strongly prefer using Method 3, each of the previous equations is most easily solved with the quadratic formula. Be sure to move all the terms to the left side of the equals sign, because the trinomial must equal 0 when it is put in standard form: $ax^2 + bx + c = 0$. When you know the equation's a, b, and c values, put them into the quadratic formula to find the two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now try to solve the following quadratic equations, using one of the four methods. Check your answers on pages 365–366.

3. $x^2 - 7 = 42$
4. $(x - 3)^2 - 5 = 11$
5. $3x^2 - 10x = 8$

Analyzing the terms of a quadratic equation is the key to choosing the method you'll use to solve it. The quadratic formula not only helps you solve complex equations, but it can also give you insight into what type of solutions an equation will have.

Finding the Number and Type of Quadratic Equation Solutions

The radical in the numerator of the quadratic formula plays a very important role in solving a quadratic equation. The radicand inside the radical is called the discriminant. Focus on that part of the formula in this next example, taken from the earlier Method 4 section:

$$6x^2 + 7x + 10 = 13$$

To find the *a*, *b*, and *c* values, move the 13 to the left side of the equals sign:

$$6x^2 + 7x + 10 - 13 = 13 - 13$$
$$6x^2 + 7x - 3 = 0$$

In this equation, $a = 6$, $b = 7$, and $c = -3$. Put these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 6 \cdot -3}}{2 \cdot 6}$$

The discriminant $b^2 - 4ac$ for this problem is $7^2 - 4 \cdot 6 \cdot -3$, which can be simplified as follows:

$$b^2 - 4ac = 7^2 - 4 \cdot 6 \cdot -3 = 49 + 72 = 121$$

JUST A MINUTE

Recall from Hour 2 that rational numbers are those that can be written as the ratio of two integers—as a fraction. The whole numbers, integers, fractions, and decimals are rational numbers.

Because 121 is a perfect square—$11 \cdot 11 = 121$—the radical in the numerator will simplify to be ±11, and there will be no square root symbol in the answer. The ± in front of the 11 shows there will be two unique solutions that are rational numbers.

Let's finish this problem to see its solutions:

$$x = \frac{-7 \pm 11}{12}$$

$$x = \frac{-7 + 11}{12} = \frac{4}{12} = \frac{1}{3} \quad \text{or} \quad x = \frac{-7 - 11}{12} = \frac{-18}{12} = -\frac{3}{2}$$

The two solutions are $x = \frac{1}{3}$ or $x = \frac{-3}{2}$. As predicted from looking at the discriminant, these are two unique rational numbers.

When the discriminant is a positive, perfect square number, the equation has two rational solutions. When it is a positive number that isn't a perfect square, there will be two irrational solutions.

In all the equations solved in the past two hours of study, the solutions have all been real numbers. This is not always the case, as you'll see in the next example's solution steps:

$$2x^2 + 3x + 4 = 0$$

This equation is already in $ax^2 + bx + c = 0$ form, so the a, b, and c values are easy to find:

Put $a = 2$, $b = 3$, and $c = 4$ into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2}$$

$$x = \frac{-3 \pm \sqrt{9 - 32}}{4}$$

$$x = \frac{-3 \pm \sqrt{-23}}{4}$$

JUST A MINUTE

Recall from Hour 21 that the square root of a negative number cannot be a real number because a number times itself cannot give a negative answer: multiplying two positive numbers gives a positive result, and multiplying two negative numbers also gives a positive. The square root of a negative number is called an imaginary number.

The square root of –23 is not a real number, since it is not possible to get a negative number when any number is multiplied times itself. When you try to simplify this radical, the result is an imaginary number, a topic you can learn more about in higher-level math courses.

For now, you have to say that the equation $2x^2 + 3x + 4 = 0$ has no real solutions.

This last example shows clearly how the discriminant can help you work more efficiently. If you had known before doing all the solution steps that there is no real solution, you could have saved time and energy for other problems.

Let's look at this equation again and check out its discriminant for helpful clues:

$$b^2 - 4ac = 3^2 - 4 \cdot 2 \cdot 4 = 9 - 32 = -23$$

TIME SAVER

You take the square root of the discriminant each time you use the quadratic formula, so the discriminant's value tells you a lot about the equation's solutions.

Seeing that the discriminant is negative, you know there are no real solutions to this quadratic equation. When the discriminant is a negative number, the equation has no real solutions.

If you begin each solution process with finding the value of the discriminant, you will know whether there are any real solutions. When the discriminant is a positive number, there will be two solutions, and they will both be real numbers, not imaginary. They could be rational—when the discriminant is a perfect square—or irrational—when it is not a perfect square.

What if the discriminant equals 0?

Look at the following quadratic equation and its discriminant.

Equation: $2x^2 + 12x + 18 = 0 \rightarrow a = 2, b = 12, c = 18$

Discriminant: $b^2 - 4ac = 12^2 - 4 \cdot 2 \cdot 18 = 144 - 144 = 0$

The square root of 0 equals 0, and ±0 is still 0. So the numerator in the quadratic formula solution for this equation is simple:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm 0}{2 \cdot 2}$$

$$x = \frac{-12 + 0}{4} = \frac{-12}{4} = -3 \quad \text{or} \quad x = \frac{-12 - 0}{4} = \frac{-12}{4} = -3$$

The solution to this equation is $x = -3$.

FYI Adding 0 to a number or subtracting 0 from a number doesn't change the number. When the discriminant equals 0, the quadratic equation has only one solution.

By finding that the value of the discriminant is 0, you can quickly see that the solution is simply $x = \frac{-b}{2a}$ because the radical part of the numerator equals 0 and only $-b$ remains in the numerator. When the discriminant equals 0, the equation has one real solution.

Review what the discriminant tells you about an equation's solutions:

Discriminant > 0 → two real solutions
Discriminant = 0 → one real solution
Discriminant < 0 → no real solutions

Now try to find the value of the discriminant for each of the following quadratic equations. Tell how many and what type of solutions it will have, checking your answers on page 366.

6. $x^2 + 4x - 12 = 0$
7. $3x^2 + 2x + 3 = 0$
8. $x^2 + 10x + 25 = 0$

Solving Practical Problems with Quadratic Equations

The applications of quadratic equations to the real world range from business to medical research, to geometry, and even to baseball, among many other areas. Your ability to solve a quadratic equation lets you participate in solving problems that are far beyond most people's basic math skills. Let's look at two examples from physics to see how the real world uses quadratic equations.

PROCEED WITH CAUTION

Choose from the four methods when solving a real-world quadratic equation. If you choose the quadratic formula, be sure to put the equation in standard from, with the terms on the left side of the equation and 0 on the right side.

As an object falls (disregarding air friction), it drops at the rate of about 32 feet per second squared. This could be written as the following quadratic equation:

$$\text{Distance} = 16x^2 \ (x \text{ is the number of seconds it falls})$$

How long will it take for the object to fall from the top of a 40-story building, assuming each story of the building is about 10 feet high?

Solution: The top of the 40-story building is about 400 feet high, $40 \cdot 10$, so 400 is the distance. The equation for this situation is:

$$400 = 16x^2$$

Solving for x tells you how many seconds it takes for the object to hit the ground. This quadratic equation is most easily solved using Method 1: finding square roots. Isolate the squared variable and take the square root of both sides of the equation, as shown here:

$$400 = 16x^2$$
$$\frac{400}{16} = \frac{16x^2}{16}$$
$$25 = x^2$$
$$\sqrt{25} = \sqrt{x^2}$$
$$\pm 5 = x$$

JUST A MINUTE

In many practical problems, only one solution is appropriate—the negative solution is often not included—even though there are two algebraic solutions to the equation.

Both +5 and –5 work in the original equation, but since x is the number of seconds, it can't be negative, so in this problem the only answer is 5. It will take about 5 seconds for the object to fall 400 feet.

Now let's look at how high a ball will be if you throw it straight up in the air at 64 feet per second, which is about 44 miles per hour. To calculate the ball's height after x seconds, use the following equation:

$$\text{Height of ball} = -16x^2 + 64x$$

Of course, the ball goes straight up, reaches its highest point above the ground, and then comes back down to you.

Question: Will the ball be 100 feet high?

Solution: Put 100 in the equation for "Height of ball" and write the new equation in standard form:

$$100 = -16x^2 + 64x$$

To put this in standard form, move the terms on the right side to the left side:

$$100 = -16x^2 + 64x$$
$$100 + 16x^2 - 64x = -16x^2 + 64x + 16x^2 - 64x$$
$$16x^2 - 64x + 100 = 0$$

In this equation, $a = 16$, $b = -64$, $c = 100$.

Put these values in the discriminant:

$$b^2 - 4ac = (-64)^2 - 4 \cdot 16 \cdot 100 = 4096 - 6400 = -2304$$

Since the discriminant is negative, there are no real solutions to the equation $100 = -16x^2 + 64x$, which is to say that the ball definitely won't reach 100 feet in height if you throw it straight up at about 44 miles per hour. If the discriminant had been zero or a positive number, the ball would have reached 100 feet of height, or higher, before returning to you.

Now try to solve this problem, checking your answer on page 366.

9. If you increase the speed you throw the ball straight up to 128 feet per second (about 88 miles per hour), will the ball reach 100 feet in the air? Use $100 = -16x^2 + 128x$ as the equation. Find the value of the discriminant to answer the question.

Answers to Sample Problems

1. $x^2 + 2x = 15 \rightarrow x^2 + 2x - 15 = 0 \rightarrow a = 1,\ b = 2,\ c = -15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot -15}}{2 \cdot 1} \rightarrow$$

$$x = \frac{-2 \pm \sqrt{4 + 60}}{2} \rightarrow x = \frac{-2 \pm \sqrt{64}}{2} \rightarrow x = \frac{-2 \pm 8}{2} \rightarrow$$

$$x = \frac{-2 + 8}{2} = \frac{6}{2} = 3 \quad \text{or} \quad x = \frac{-2 - 8}{2} = \frac{-10}{2} = -5$$

2. $6x^2 = -5x + 4 \rightarrow 6x^2 + 5x - 4 = 0 \rightarrow a = 6,\ b = 5,\ c = -4 \rightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot -4}}{2 \cdot 6} \rightarrow$$

$$x = \frac{-5 \pm \sqrt{25 + 96}}{12} \rightarrow x = \frac{-5 \pm \sqrt{121}}{12} \rightarrow x = \frac{-5 \pm 11}{12} \rightarrow$$

$$x = \frac{-5 + 11}{12} = \frac{6}{12} = \frac{1}{2} \quad \text{or} \quad x = \frac{-5 - 11}{12} = \frac{-16}{12} = \frac{-4}{3}$$

3. $x^2 - 7 = 42 \rightarrow$ Finding square roots $\rightarrow x^2 - 7 + 7 = 42 + 7 \rightarrow$

$x^2 = 49 \rightarrow \sqrt{x^2} = \sqrt{49} \rightarrow x = \pm 7 \rightarrow x = 7 \quad \text{or} \quad x = -7$

4. $(x-3)^2 - 5 = 11 \rightarrow$ Finding square roots $\rightarrow (x-3)^2 - 5 + 5 = 11 + 5 \rightarrow (x-3)^2 = 16 \rightarrow \sqrt{(x-3)^2} = \sqrt{16} \rightarrow x - 3 = \pm 4 \rightarrow x - 3 + 3 = \pm 4 + 3 \rightarrow x = 3 \pm 4 \rightarrow x = 3 + 4 = 7$ or $x = 3 - 4 = -1$

5. $3x^2 - 10x = 8 \rightarrow$ Quadratic formula $\rightarrow 3x^2 - 10x - 8 = 0 \rightarrow a = 3,\ b = -10,\ c = -8 \rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \dfrac{10 \pm \sqrt{(-10)^2 - 4 \cdot 3 \cdot -8}}{2 \cdot 3} \rightarrow x = \dfrac{10 \pm \sqrt{100 + 96}}{6} \rightarrow x = \dfrac{10 \pm \sqrt{196}}{6} \rightarrow x = \dfrac{10 \pm 14}{6} \rightarrow x = \dfrac{10 + 14}{6} = \dfrac{24}{6} = 4$ or $x = \dfrac{10 - 14}{6} = \dfrac{-4}{6} = \dfrac{-2}{3}$

6. $x^2 + 4x - 12 = 0 \rightarrow a = 1,\ b = 4,\ c = -12 \rightarrow b^2 - 4ac \rightarrow 4^2 - 4 \cdot 1 \cdot -12 = 16 + 48 = 64$
   Since the discriminant is a positive perfect square, there will be two rational solutions.

7. $3x^2 + 2x + 3 = 0 \rightarrow a = 3,\ b = 2,\ c = 3 \rightarrow b^2 - 4ac \rightarrow 2^2 - 4 \cdot 3 \cdot 3 = 4 - 36 = -32$
   Since the discriminant is negative, there are no real solutions.

8. $x^2 + 10x + 25 = 0 \rightarrow a = 1,\ b = 10,\ c = 25 \rightarrow b^2 - 4ac \rightarrow 10^2 - 4 \cdot 1 \cdot 25 = 100 - 100 = 0$
   Since the discriminant equals 0, there is one real solution.

9. $100 = -16x^2 + 128x$
   Move the terms to the left side, so the right side is 0:
   $100 = -16x^2 + 128x \rightarrow 16x^2 - 128x + 100 = 0 \rightarrow a = 16,\ b = -128,\ c = 100 \rightarrow b^2 - 4ac \rightarrow (-128)^2 - 4 \cdot 16 \cdot 100 = 16384 - 6400 = 9984$
   Since the discriminant is positive, the ball will reach 100 feet in height—on the way up to its final height, and also on the way back down.

Hour's Up!

Based on what you've learned in Hour 24, solve the following quadratic equations using one of the four methods. Check your answers with the solution key in Appendix B.

1. $x^2 = 32$
- **a.** $x = 16$ or -16
- **b.** $x = 4$ or -4
- **c.** $x = 4\sqrt{2}$ or $-4\sqrt{2}$
- **d.** $x = 3\sqrt{2}$ or $-3\sqrt{2}$

2. $x^2 + 10 = 46$
- **a.** $x = 36$ or -36
- **b.** $x = 6$ or -6
- **c.** $x = 5$ or -5
- **d.** $x = 4$ or -4

3. $x^2 - 8 = 92$
- **a.** $x = 10$ or -10
- **b.** $x = 8$ or -8
- **c.** $x = 6$ or -6
- **d.** $x = 5$ or -5

4. $(x - 2)^2 = 49$
- **a.** $x = 9$ or 5
- **b.** $x = -9$ or 5
- **c.** $x = 9$ or -5
- **d.** $x = -9$ or 5

5. $3(x - 4)^2 = 27$
- **a.** $x = -7$ or -1
- **b.** $x = 7$ or 1
- **c.** $x = 7$ or -1
- **d.** $x = -7$ or 1

6. $x^2 - 7x = 18$
- **a.** $x = 9$ or -2
- **b.** $x = -9$ or 2
- **c.** $x = 9$ or 2
- **d.** $x = -9$ or -2

7. $10x^2 = 13x + 3$

a. $x = \frac{3}{2}$ or $\frac{1}{5}$

b. $x = -\frac{3}{2}$ or $\frac{1}{5}$

c. $x = \frac{3}{2}$ or $-\frac{1}{5}$

d. $x = -\frac{3}{2}$ or $-\frac{1}{5}$

8. $2x^2 + 7x = 4$

a. $x = -4$ or $\frac{1}{2}$

b. $x = 4$ or $\frac{1}{2}$

c. $x = -4$ or $-\frac{1}{2}$

d. $x = 4$ or $-\frac{1}{2}$

9. Does $x^2 + 6x - 2 = 0$ have any real solutions?

a. Yes, one rational solution

b. Yes, two rational solutions

c. Yes, three irrational solutions

d. No real solutions

10. Does $4x^2 + 12x + 9 = 0$ have any real solutions?

a. Yes, one rational solution

b. Yes, two rational solutions

c. Yes, three irrational solutions

d. No real solutions

Review

Appendix A

Glossary

absolute value A number's distance from zero on the number line.

addition property of equality Adding the same number or term to both sides of a true equality creates a new equation that is also true.

associative property Terms can be grouped in any order for addition or multiplication.

base The number or variable in an exponential expression that is raised to a power or exponent (in x^3, x is the base and 3 is the power or exponent).

binomial A polynomial with two terms. Examples are $x + y$; $2a - 5$; $x^2 + 7$.

coefficient The number in front of the variable. For example, in $3x$, 3 is the coefficient.

common denominator The number or variable(s) that are the same denominator in two or more fractions, so that the fractions can be added or subtracted.

common factors Numbers or variables that are divisible into another number or variable. For example, 4 is a common factor of 12 and 16; 3 is not a common factor of 12 and 16; x is a common factor of xy and xz; and x is not a common factor of xy and yz.

commutative property Two numbers or terms can be added or multiplied in either order. For example, $2 + 3 = 3 + 2$; $x \cdot y = y \cdot x$.

completing the square One of four processes used to solve some quadratic equations by making the quadratic trinomial factorable into a squared binomial.

consecutive integers Integers that are next to each other on the real number line. For example, 2 and 3 are consecutive integers, as are –8 and –7.

denominator The part of a fraction under the fraction line.

direct variation A relationship shown by $y = kx$, where k is the constant rate of change and, as x increases, y also increases, or as x decreases, y also decreases (it is the same as directly proportional).

discriminant The radicand from the quadratic formula: $b^2 - 4ac$.

distributive property Multiplying a sum by a number or variable is equal to the sum of the product of the numbers. For example, $2(x + y) = 2x + 2y$.

division property of equality Both sides of an equation can be divided by the same nonzero number or variable, and the new equation will be true.

equation An expression that includes an equal sign. For example, $2 + 3 = 5$ and $x + 2y = -4$.

evaluate To substitute numbers for variables in an expression and calculate a numerical answer.

expanded form A monomial written with no exponents. For example, $x \cdot x \cdot x$ is the expanded form of x^3.

exponent The number (power) that is above and to the right of the base. For example, in x^3, 3 is the exponent and indicates how many times to use the base x as a factor.

factoring Showing the numbers or variables that multiply to form another number or monomial term. For example, factoring $9x$ gives $3 \cdot 3 \cdot x$.

factors Numbers or variables that can be multiplied to give a number or monomial term.

FOIL method A shortcut way to multiply two binomials.

graphing Putting points on a number line or ordered pairs on a coordinate system of x- and y-axes.

grouping symbols Parentheses or brackets used to separate or emphasize two or more numbers or variables.

identity property of addition Adding 0 to any number or variable does not change the number or the variable.

identity property of multiplication Multiplying a number or variable by 1 does not change the number or variable.

imaginary number A number that is the result of taking the square root of a negative number.

infinite Continuing without end; the integers are infinite because there is always another integer larger than or smaller than any integer.

integers The set of whole numbers and their opposites (…, –2, –1, 0, 1, 2, …).

inverse variation A relationship shown by $xy = k$, where k is the constant rate of change, and as x increases, y decreases, or as x decreases, y increases.

irrational numbers Numbers that are the result of taking the square root of a number that has a factor that is not a perfect square ($\sqrt{8}$ is irrational; $\sqrt{9} = 3$ is rational).

isolate the variable To solve an equation for the variable.

monomial A polynomial with one term. Examples are $4ab$, 4, ab, a, b, and b^3.

multiplication property of equality Both sides of an equation can be multiplied by the same number or variable, and the new equation will be true.

numerator The part of a fraction above the fraction line.

open equations Equations that might not be true, depending on what value is given to the variable. For example, $2x + y = 6$ and $x + 7 = -5$ are open equations.

operations Adding, subtracting, multiplying, dividing, squaring, and taking the square root of numbers or variables.

opposites Numbers or variables that have different signs (2 and –2 are opposites; x and –x are opposites).

opposites property of addition When two opposites are added, the result is 0.

ordered pair Two numbers that represent a solution to a two-variable equation; (1,2) is not the same as (2,1) because the order of the numbers is different.

ordering numbers Comparing two or more numbers to list them in ascending or descending order.

origin On an (x,y) graph, the point (0,0); on a number line, the point 0.

parallel lines Lines in the same plane that never intersect.

parentheses Can be used as grouping symbols: $7 \cdot (2 + 3x)$; to show multiplication: $(7)(5) = 35$; to show emphasis $(-3) + 7 = 4$; or to raise a power to a power: $(2^4)^3 = 2^{12}$.

percent From "per hundred"; for example, 6/100 = 6% = .06.

perfect squares Numbers that come from multiplying a number times itself (1, 4, 9, 16, 25, …).

plotting Graphing points on a number line or ordered pairs on an (x,y) axis graph. See also: monomials, binomials, trinomials.

polynomials Numbers or variables that are added or subtracted to form an expression. For example, $2c$; $3x - 9$; and $5abc + 8$ are all polynomials. See also: monomials, binomials, trinomials.

power Exponent or the number that indicates how many times to use a base as a factor; for example, 4^3 has a power of 3 and shows that 4 is to be multiplied times itself 3 times.

proportion Two ratios (or fractions) that are equal.

quadratic equation An equation with a variable with an exponent of 2.

quadratic formula A method of solving a quadratic equation of the form $ax^2 + bx + c = 0$, where $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

radical Square root.

radicand The number or variable(s) inside the square root symbol.

ratio A fraction.

rational number A number that can be written as the ratio of two integers.

real number line The horizontal line that contains marked intervals evenly spaced, with the negative numbers on the left and positive numbers on the right.

real numbers The numbers that can be plotted on the real number line (all numbers are real except the square root of a negative number).

reciprocal A number that, when multiplied by another number, gives a product of 1.

reciprocal property of multiplication Two reciprocals have a product of 1.

rise In slope, the rise is the change in the y values for two ordered pairs.

run In slope, the run is the change in the x values for two ordered pairs.

simplifying Reducing a fraction to its lowest terms or combining like terms in an expression.

simplifying radicals Removing all perfect square factors from the radicand.

slope The ratio of rise to run of a line.

slope-intercept formula The formula $y = mx + b$, where m is the slope and b is the y-intercept of the line on an (x,y) graph.

subtraction property of equality A number or variable can be subtracted from both sides of an equation, and the new equation will be true.

systems of equations Two equations with two of the same variables in each equation ($x + y = 6$ and $2x + y = -8$ form a system of equations).

trinomial A polynomial with three monomials. For example, $x + y - z$ and $2a + 4c + t$.

variables Letters that represent numbers in an expression.

***x*-intercept** The point where a graphed line crosses the x-axis of the (x,y) coordinate graph (the ordered pair where $y = 0$).

***y*-intercept** The point where a graphed line crosses the y-axis of the (x,y) coordinate graph (the ordered pair where $x = 0$).

Appendix B

Hour's Up! Answers

Hour 1

1. c
2. b
3. d
4. b
5. a
6. c
7. a
8. b
9. c
10. b

Hour 2

1. c
2. c
3. b
4. d
5. d
6. b
7. b
8. d
9. b
10. a

Hour 3

1. c
2. a
3. d
4. c
5. a
6. b
7. c
8. d
9. b
10. c

Hour 4

1. c
2. b
3. a
4. a
5. d
6. c
7. a
8. c
9. a
10. c

Hour 5

1. d
2. b
3. a
4. d
5. a
6. b
7. c
8. d
9. d
10. a

Hour 6

1. d
2. b
3. c
4. a
5. b
6. a
7. b
8. a
9. c
10. d

Hour 7

1. d
2. a
3. c
4. d
5. a
6. b
7. c
8. b
9. c
10. a

Hour 8

1. c
2. b
3. d
4. a
5. c
6. a
7. b
8. a
9. b
10. c

Hour 9

1. b
2. a
3. d
4. d
5. b
6. c
7. a
8. b
9. d
10. c

Hour 10

1. c
2. b
3. d
4. c
5. d
6. a
7. b
8. a
9. c
10. d

Hour 11

1. d
2. a
3. b
4. a
5. d
6. b
7. c
8. d
9. c
10. a

Hour 12

1. b
2. d
3. a
4. d
5. b
6. c
7. c
8. d
9. a
10. b

Hour 13

1. c
2. a
3. d
4. c
5. d
6. b
7. a
8. d
9. a
10. b

Hour 14

1. b
2. d
3. a
4. c
5. a
6. d
7. a
8. b
9. c
10. a

Hour 15

1. c
2. b
3. c
4. a
5. b
6. c
7. a
8. d
9. c
10. d

Hour 16

1. b
2. c
3. d
4. c
5. a
6. a
7. b
8. d
9. a
10. b

Hour 17

1. b
2. d
3. b
4. a
5. d
6. c
7. a
8. d
9. b
10. a

Hour 18

1. c
2. b
3. c
4. b
5. a
6. c
7. b
8. c
9. b
10. a

Hour 19

1. a
2. d
3. c
4. a
5. b
6. a
7. c
8. d
9. b
10. a

Hour 20

1. c
2. b
3. d
4. c
5. b
6. a
7. c
8. a
9. d
10. c

Hour 21

1. a
2. b
3. d
4. d
5. c
6. a
7. b
8. c
9. d
10. b

Hour 22

1. b
2. d
3. c
4. c
5. d
6. b
7. d
8. a
9. b
10. c

Hour 23

1. c
2. b
3. c
4. a
5. b
6. c
7. d
8. a
9. d
10. b

Hour 24

1. c
2. b
3. a
4. c
5. b
6. a
7. c
8. a
9. c
10. a

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