

APPENDIX

Solutions to Selected Exercises

CHAPTER 1

- 1.3.** $A \cup C \subset B$ implies $A \subset B$. Hence, $A = A \cap B$. Thus, $A \cap B \subset \bar{C}$ implies $A \subset \bar{C}$. It follows that $A \cap C = \emptyset$.
- 1.4.** (a) $x \in A \Delta B$ implies that $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$; thus $x \in A \cup B - A \cap B$. Vice versa, if $x \in A \cup B - A \cap B$, then $x \in A \Delta B$.
- (c) $x \in A \cap (B \Delta D)$ implies that $x \in A$ and $x \in B \Delta D$, so that either $x \in A \cap B$ but $x \notin A \cap D$, or $x \in A \cap D$ but $x \notin A \cap B$, so that $x \in (A \cap B) \Delta (A \cap D)$. Vice versa, if $x \in (A \cap B) \Delta (A \cap D)$, then either $x \in A \cap B$ but $x \notin A \cap D$, or $x \in A \cap D$ but $x \notin A \cap B$, so that either x is in A and B but $x \notin D$, or x is in A and D , but $x \notin B$; thus $x \in A \cap (B \Delta D)$.
- 1.5.** It is obvious that ρ is reflexive, symmetric, and transitive. Hence, it is an equivalence relation. If (m_0, n_0) is an element in A , then its equivalence class is the set of all pairs (m, n) in A such that $m/m_0 = n/n_0$, that is, $m/n = m_0/n_0$.
- 1.6.** The equivalence class of $(1, 2)$ consists of all pairs (m, n) in A such that $m - n = -1$.
- 1.7.** The first elements in all four pairs are distinct.
- 1.8.** (a) If $y \in f(\bigcup_{i=1}^n A_i)$, then $y = f(x)$, where $x \in \bigcup_{i=1}^n A_i$. Hence, if $x \in A_i$ and $f(x) \in f(A_i)$ for some i , then $f(x) \in \bigcup_{i=1}^n f(A_i)$; thus $f(\bigcup_{i=1}^n A_i) \subset \bigcup_{i=1}^n f(A_i)$. Vice versa, it is easy to show that $\bigcup_{i=1}^n f(A_i) \subset f(\bigcup_{i=1}^n A_i)$.
- (b) If $y \in f(\bigcap_{i=1}^n A_i)$, then $y = f(x)$, where $x \in A_i$ for all i ; then $f(x) \in f(A_i)$ for all i ; then $f(x) \in \bigcap_{i=1}^n f(A_i)$. Equality holds if f is one-to-one.

- 1.11.** Define $f: J^+ \rightarrow A$ such that $f(n) = 2n^2 + 1$. Then f is one-to-one and onto, so A is countable.
- 1.13.** $a + \sqrt{b} = c + \sqrt{d} \Rightarrow a - c = \sqrt{d} - \sqrt{b}$. If $a = c$, then $b = d$. If $a \neq c$, then $\sqrt{d} - \sqrt{b}$ is a nonzero rational number and $\sqrt{d} + \sqrt{b} = (d - b)/(\sqrt{d} - \sqrt{b})$. It follows that both $\sqrt{d} - \sqrt{b}$ and $\sqrt{d} + \sqrt{b}$ are rational numbers, and therefore $\sqrt{b}$ and $\sqrt{d}$ must be rational numbers.
- 1.14.** Let $g = \inf(A)$. Then $g \leq x$ for all x in A , so $-g \geq -x$, hence, $-g$ is an upper bound of $-A$ and is the least upper bound: if $-g'$ is any other upper bound of $-A$, then $-g' \geq -x$, so $x \geq g'$, hence, g' is a lower bound of A , so $g' \leq g$, that is, $-g' \geq -g$, so $-g = \sup(-A)$.
- 1.15.** Suppose that $b \notin A$. Since b is the least upper bound of A , it must be a limit point of A : for any $\epsilon > 0$, there exists an element $a \in A$ such that $a > b - \epsilon$. Furthermore, $a < b$, since $b \notin A$. Hence, b is a limit point of A . But A is closed; hence, by Theorem 1.6.4, $b \in A$.
- 1.17.** Suppose that G is a basis for $\mathcal{F}$, and let $p \in B$. Then $B = \bigcup_{\alpha} U_{\alpha}$, where U_{α} belongs to G . Hence, there is at least one U_{α} such that $p \in U_{\alpha} \subset B$. Vice versa, if for each $B \in \mathcal{F}$ and each $p \in B$ there is a $U \in G$ such that $p \in U \subset B$, then G must be a basis for $\mathcal{F}$: for each $p \in B$ we can find a set $U_p \in G$ such that $p \in U_p \subset B$; then $B = \bigcup \{U_p | p \in B\}$, so G is a basis.
- 1.18.** Let p be a limit point of $A \cup B$. Then p is a limit point of A or of B . In either case, $p \in A \cup B$. Hence, by Theorem 1.6.4, $A \cup B$ is a closed set.
- 1.19.** Let $\{C_{\alpha}\}$ be an open covering of B . Then $\bar{B}$ and $\{C_{\alpha}\}$ form an open covering of A . Since A is compact, a finite subcollection of the latter covering covers A . Furthermore, since $\bar{B}$ does not cover B , the members of this finite covering that contain B are all in $\{C_{\alpha}\}$. Hence, B is compact.
- 1.20.** No. Let $(A, \mathcal{F})$ be a topological space such that A consists of two points a and b and $\mathcal{F}$ consists of A and the empty set $\emptyset$. Then A is compact, and the point a is a compact subset; but it is not closed, since the complement of a , namely b , is not a member of $\mathcal{F}$, and is therefore not open.
- 1.21. (a)** Let $w \in A$. Then $X(w) \leq x < x + 3^{-n}$ for all n , so $w \in \bigcap_{n=1}^{\infty} A_n$. Vice versa, if $w \in \bigcap_{n=1}^{\infty} A_n$, then $X(w) < x + 3^{-n}$ for all n . To show that $X(w) \leq x$: if $X(w) > x$, then $X(w) > x + 3^{-n}$ for some n , a contradiction. Therefore, $X(w) \leq x$, and $w \in A$.

- (b) Let $w \in B$. Then $X(w) < x$, so $X(w) < x - 3^{-n}$ for some n , hence, $w \in \bigcup_{n=1}^{\infty} B_n$. Vice versa, if $w \in \bigcup_{n=1}^{\infty} B_n$, then $X(w) \leq x - 3^{-n}$ for some n , so $X(w) < x$; if $X(w) \geq x$, then $X(w) > x - 3^{-n}$ for all n , a contradiction. Therefore, $w \in B$.

1.22. (a) $P(X \geq 2) \leq \lambda/2$, since $\mu = \lambda$.

(b)

$$\begin{aligned} P(X \geq 2) &= 1 - p(X < 2) \\ &= 1 - p(0) - p(1) \\ &= 1 - e^{-\lambda}(\lambda + 1). \end{aligned}$$

To show that

$$1 - e^{-\lambda}(\lambda + 1) < \frac{\lambda}{2}.$$

Let $\phi(\lambda) = \lambda/2 + e^{-\lambda}(\lambda + 1) - 1$. Then

- i. $\phi(0) = 0$, and
- ii. $\phi'(\lambda) = \frac{1}{2} - \lambda e^{-\lambda} > 0$ for all λ .

From (i) and (ii) we conclude that $\phi(\lambda) > 0$ for all $\lambda > 0$.

1.23. (a)

$$\begin{aligned} p(|X - \mu| \geq c) &= P[(X - \mu)^2 \geq c^2] \\ &\leq \frac{\sigma^2}{c^2}, \end{aligned}$$

by Markov's inequality and the fact that $E(X - \mu)^2 = \sigma^2$.

(b) Use $c = k\sigma$ in (a).

(c)

$$\begin{aligned} P(|X - \mu| < k\sigma) &= 1 - P(|X - \mu| \geq k\sigma) \\ &\geq 1 - \frac{1}{k^2}. \end{aligned}$$

1.24. $\mu = E(X) = \int_{-1}^1 x(1 - |x|) dx = 0$, $\sigma^2 = \int_{-1}^1 x^2(1 - |x|) dx = \frac{1}{6}$.

(a)

- i. $P(|X| \geq \frac{1}{2}) \leq \sigma^2 / \frac{1}{4} = \frac{2}{3}$.
- ii. $P(|X| > \frac{1}{3}) = P(|X| \geq \frac{1}{3}) \leq \sigma^2 / \frac{1}{9} = \frac{3}{2}$.

(b)

$$\begin{aligned} P(|X| \geq \tfrac{1}{2}) &= P(X \geq \tfrac{1}{2}) + P(X \leq -\tfrac{1}{2}) \\ &= \tfrac{1}{4} < \tfrac{2}{3}. \end{aligned}$$

- 1.25. (a) True, since $X_{(1)} \geq x$ if and only if $X_i \geq x$ for all i .
 (b) True, since $X_{(n)} \leq x$ if and only if $X_i \leq x$ for all i .
 (c)

$$\begin{aligned} P(X_{(1)} \leq x) &= 1 - P(X_{(1)} > x) \\ &= 1 - [1 - F(x)]^n. \end{aligned}$$

$$(d) P(X_{(n)} \leq x) = [F(x)]^n.$$

- 1.26. $P(2 \leq X_{(1)} \leq 3) = P(X_{(1)} \leq 3) - P(X_{(1)} \leq 2)$. Hence,

$$\begin{aligned} P(2 \leq X_{(1)} \leq 3) &= 1 - [1 - F(3)]^5 - 1 + [1 - F(2)]^5 \\ &= [1 - F(2)]^5 - [1 - F(3)]^5. \end{aligned}$$

But $F(x) = \int_0^x 2e^{-2t} dt = 1 - e^{-2x}$. Hence,

$$\begin{aligned} P(2 \leq X_{(1)} \leq 3) &= (e^{-4})^5 - (e^{-6})^5 \\ &= e^{-20} - e^{-30}. \end{aligned}$$

CHAPTER 2

- 2.1. If $m > n$, then the rank of the $n \times m$ matrix $\mathbf{U} = [\mathbf{u}_1 : \mathbf{u}_2 : \cdots : \mathbf{u}_m]$ is less than or equal to n . Hence, the number of linearly independent columns of $\mathbf{U}$ is less than or equal to n , so the m columns of $\mathbf{U}$ must be linearly dependent.
- 2.2. If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ and $\mathbf{v}$ are linearly dependent, then $\mathbf{v}$ must belong to W , a contradiction.
- 2.5. Suppose that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent in U , and that $\sum_{i=1}^n \alpha_i T(\mathbf{u}_i) = \mathbf{0}$ for some constants $\alpha_1, \alpha_2, \dots, \alpha_n$. Then $T(\sum_{i=1}^n \alpha_i \mathbf{u}_i) = \mathbf{0}$. If T is one-to-one, then $\sum_{i=1}^n \alpha_i \mathbf{u}_i = \mathbf{0}$, hence, $\alpha_i = 0$ for all i , since the $\mathbf{u}_i$'s are linearly independent. It follows that $T(\mathbf{u}_1), T(\mathbf{u}_2), \dots, T(\mathbf{u}_n)$ must also be linearly independent. Vice versa, let $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{0}$. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be a basis for U . Then $\mathbf{u} = \sum_{i=1}^n \tau_i \mathbf{e}_i$ for some constants $\tau_1, \tau_2, \dots, \tau_n$. $T(\mathbf{u}) = \mathbf{0} \Rightarrow \sum_{i=1}^n \tau_i T(\mathbf{e}_i) = \mathbf{0} \Rightarrow \tau_i = 0$ for all i , since $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$ are linearly independent. It follows that $\mathbf{u} = \mathbf{0}$ and T is one-to-one.
- 2.7. If $\mathbf{A} = (a_{ij})$, then $\text{tr}(\mathbf{A}\mathbf{A}) = \sum_i \sum_j a_{ij}^2$. Hence, $\mathbf{A} = \mathbf{0}$ if and only if $\text{tr}(\mathbf{A}\mathbf{A}) = 0$.

2.8. It is sufficient to show that $\mathbf{A}\mathbf{v} = \mathbf{0}$ if $\mathbf{v}'\mathbf{A}\mathbf{v} = 0$. If $\mathbf{v}'\mathbf{A}\mathbf{v} = \mathbf{0}$, then $\mathbf{A}^{1/2}\mathbf{v} = \mathbf{0}$, and hence $\mathbf{A}\mathbf{v} = \mathbf{0}$. [Note: $\mathbf{A}^{1/2}$ is defined as follows: since $\mathbf{A}$ is symmetric, it can be written as $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$ by Theorem 2.3.10, where $\mathbf{\Lambda}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$, and $\mathbf{P}$ is an orthogonal matrix. Furthermore, since $\mathbf{A}$ is positive semidefinite, its eigenvalues are greater than or equal to zero by Theorem 2.3.13. The matrix $\mathbf{A}^{1/2}$ is defined as $\mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'$, where the diagonal elements of $\mathbf{\Lambda}^{1/2}$ are the square roots of the corresponding diagonal elements of $\mathbf{\Lambda}$.]

2.9. By Theorem 2.3.15, there exists an orthogonal matrix $\mathbf{P}$ and diagonal matrices $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$ such that $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}_1\mathbf{P}'$, $\mathbf{B} = \mathbf{P}\mathbf{\Lambda}_2\mathbf{P}'$. The diagonal elements of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are nonnegative. Hence,

$$\mathbf{AB} = \mathbf{P}\mathbf{\Lambda}_1\mathbf{\Lambda}_2\mathbf{P}'$$

is positive semidefinite, since the diagonal elements of $\mathbf{\Lambda}_1\mathbf{\Lambda}_2$ are nonnegative.

2.10. Let $\mathbf{C} = \mathbf{AB}$. Then $\mathbf{C}' = -\mathbf{BA}$. Since $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, we have $\text{tr}(\mathbf{C}) = \text{tr}(-\mathbf{C}') = -\text{tr}(\mathbf{C})$ and thus $\text{tr}(\mathbf{C}) = 0$.

2.11. Let $\mathbf{B} = \mathbf{A} - \mathbf{A}'$. Then

$$\begin{aligned} \text{tr}(\mathbf{B}'\mathbf{B}) &= \text{tr}[(\mathbf{A}' - \mathbf{A})(\mathbf{A} - \mathbf{A}')] \\ &= \text{tr}[(\mathbf{A}' - \mathbf{A})\mathbf{A}] - \text{tr}[(\mathbf{A}' - \mathbf{A})\mathbf{A}'] \\ &= \text{tr}[(\mathbf{A}' - \mathbf{A})\mathbf{A}] - \text{tr}[\mathbf{A}(\mathbf{A} - \mathbf{A}')] \\ &= \text{tr}[(\mathbf{A}' - \mathbf{A})\mathbf{A}] - \text{tr}[(\mathbf{A} - \mathbf{A}')\mathbf{A}] \\ &= 2\text{tr}[(\mathbf{A}' - \mathbf{A})\mathbf{A}] = 0, \end{aligned}$$

since $\mathbf{A}' - \mathbf{A}$ is skew-symmetric. Hence, by Exercise 2.7, $\mathbf{B} = \mathbf{0}$ and thus $\mathbf{A} = \mathbf{A}'$.

2.12. This follows easily from Theorem 2.3.9.

2.13. By Theorem 2.3.10, we can write

$$\mathbf{A} - \lambda\mathbf{I}_n = \mathbf{P}(\mathbf{\Lambda} - \lambda\mathbf{I}_n)\mathbf{P}',$$

where $\mathbf{P}$ is an orthogonal matrix and $\mathbf{\Lambda}$ is diagonal with diagonal elements equal to the eigenvalues of $\mathbf{A}$. The diagonal elements of $\mathbf{\Lambda} - \lambda\mathbf{I}_n$ are the eigenvalues of $\mathbf{A} - \lambda\mathbf{I}_n$. Now, k diagonal elements of $\mathbf{\Lambda} - \lambda\mathbf{I}_n$ are equal to zero, and the remaining $n - k$ elements must be different from zero. Hence, $\mathbf{A} - \lambda\mathbf{I}_n$ has rank $n - k$.

- 2.17.** If $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$, then $(\mathbf{A} - \mathbf{B})^2 = \mathbf{A}^2 - \mathbf{AB} - \mathbf{BA} + \mathbf{B}^2 = \mathbf{A} - 2\mathbf{B} + \mathbf{B} = \mathbf{A} - \mathbf{B}$. Vice versa, if $\mathbf{A} - \mathbf{B}$ is idempotent, then $\mathbf{A} - \mathbf{B} = (\mathbf{A} - \mathbf{B})^2 = \mathbf{A}^2 - \mathbf{AB} - \mathbf{BA} + \mathbf{B}^2 = \mathbf{A} - \mathbf{AB} - \mathbf{BA} + \mathbf{B}$. Thus,

$$\mathbf{AB} + \mathbf{BA} = 2\mathbf{B},$$

so $\mathbf{AB} + \mathbf{ABA} = 2\mathbf{AB}$ and thus $\mathbf{AB} = \mathbf{ABA}$. We also have

$$\mathbf{ABA} + \mathbf{BA} = 2\mathbf{BA}, \quad \text{hence, } \mathbf{BA} = \mathbf{ABA}.$$

It follows that $\mathbf{AB} = \mathbf{BA}$. We finally conclude that

$$\mathbf{B} = \mathbf{AB} = \mathbf{BA}.$$

- 2.18.** If $\mathbf{A}$ is an $n \times n$ orthogonal matrix with determinant 1, then its eigenvalues are of the form $e^{\pm i\phi_1}, e^{\pm i\phi_2}, \dots, e^{\pm i\phi_q}, 1$, where 1 is of multiplicity $n - 2q$ and none of the real numbers ϕ_j is a multiple of 2π ($j = 1, 2, \dots, q$) (those that are odd multiples of π give an even number of eigenvalues equal to -1).

- 2.19.** Using Theorem 2.3.10, it is easy to show that

$$e_{\min}(\mathbf{A})\mathbf{I}_n \leq \mathbf{A} \leq e_{\max}(\mathbf{A})\mathbf{I}_n,$$

where the inequality on the right means that $e_{\max}(\mathbf{A})\mathbf{I}_n - \mathbf{A}$ is positive semidefinite, and the one on the left means that $\mathbf{A} - e_{\min}(\mathbf{A})\mathbf{I}_n$ is positive semidefinite. It follows that

$$e_{\min}(\mathbf{A})\mathbf{L}'\mathbf{L} \leq \mathbf{L}'\mathbf{A}\mathbf{L} \leq e_{\max}(\mathbf{A})\mathbf{L}'\mathbf{L}$$

Hence, by Theorem 2.3.19(1),

$$e_{\min}(\mathbf{A}) \operatorname{tr}(\mathbf{L}'\mathbf{L}) \leq \operatorname{tr}(\mathbf{L}'\mathbf{A}\mathbf{L}) \leq e_{\max}(\mathbf{A}) \operatorname{tr}(\mathbf{L}'\mathbf{L}).$$

- 2.20.** (a) We have that $\mathbf{A} \geq e_{\min}(\mathbf{A})\mathbf{I}_n$ by Theorem 2.3.10. Hence, $\mathbf{L}'\mathbf{A}\mathbf{L} \geq e_{\min}(\mathbf{A})\mathbf{L}'\mathbf{L}$, and therefore, $e_{\min}(\mathbf{L}'\mathbf{A}\mathbf{L}) \geq e_{\min}(\mathbf{A})e_{\min}(\mathbf{L}'\mathbf{L})$ by Theorem 2.3.18 and the fact that $e_{\min}(\mathbf{A}) \geq 0$ and $\mathbf{L}'\mathbf{L}$ is nonnegative definite.
(b) This is similar to (a).

- 2.21.** We have that

$$e_{\min}(\mathbf{B})\mathbf{A} \leq \mathbf{A}^{1/2}\mathbf{B}\mathbf{A}^{1/2} \leq e_{\max}(\mathbf{B})\mathbf{A},$$

since $\mathbf{A}$ is nonnegative definite. Hence,

$$e_{\min}(\mathbf{B}) \operatorname{tr}(\mathbf{A}) \leq \operatorname{tr}(\mathbf{A}^{1/2}\mathbf{B}\mathbf{A}^{1/2}) \leq e_{\max}(\mathbf{B}) \operatorname{tr}(\mathbf{A}).$$

The result follows by noting that

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{A}^{1/2} \mathbf{BA}^{1/2}).$$

- 2.23. (a)** $\mathbf{A} = \mathbf{I}_n - (1/n)\mathbf{J}_n$, where $\mathbf{J}_n$ is the matrix of ones of order $n \times n$. $\mathbf{A}^2 = \mathbf{A}$, since $\mathbf{J}_n^2 = n\mathbf{J}_n$. The rank of $\mathbf{A}$ is the same as its trace, which is equal to $n - 1$.
- (b)** $(n - 1)s^2/\sigma^2 = (1/\sigma^2)\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi_{n-1}^2$, since $(1/\sigma^2)\mathbf{A}(\sigma^2\mathbf{I}_n)$ is idempotent of rank $n - 1$, and the noncentrality parameter is zero.
- (c)**

$$\begin{aligned} \text{Cov}(\bar{y}, \mathbf{A}\mathbf{y}) &= \text{Cov}\left(\frac{1}{n}\mathbf{1}'_n\mathbf{y}, \mathbf{A}\mathbf{y}\right) \\ &= \frac{1}{n}\mathbf{1}'_n(\sigma^2\mathbf{I}_n)\mathbf{A} \\ &= \frac{\sigma^2}{n}\mathbf{1}'_n\mathbf{A} \\ &= \mathbf{0}'. \end{aligned}$$

Since $\mathbf{y}$ is normally distributed, both $\bar{y}$ and $\mathbf{A}\mathbf{y}$, being linear transformations of $\mathbf{y}$, are also normally distributed. Hence, they must be independent, since they are uncorrelated. We can similarly show that $\bar{y}$ and $\mathbf{A}^{1/2}\mathbf{y}$ are uncorrelated, and hence independent, by the fact that $\mathbf{1}'_n\mathbf{A} = \mathbf{0}'$ and thus $\mathbf{1}'_n\mathbf{A}\mathbf{1}_n = 0$, which means $\mathbf{1}'_n\mathbf{A}^{1/2} = \mathbf{0}'$. It follows that $\bar{y}$ and $\mathbf{y}'\mathbf{A}^{1/2}\mathbf{A}^{1/2}\mathbf{y} = \mathbf{y}'\mathbf{A}\mathbf{y}$ are independent.

- 2.24. (a)** The model is written as

$$\mathbf{y} = \mathbf{X}\mathbf{g} + \boldsymbol{\epsilon},$$

where $\mathbf{X} = [\mathbf{1}_N : \bigoplus_{i=1}^a \mathbf{1}_{n_i}]$, $\bigoplus$ denotes the direct sum of matrices, $\mathbf{g} = (\mu, \alpha_1, \alpha_2, \dots, \alpha_a)'$, and $N = \sum_{i=1}^a n_i$. Now, $\mu + \alpha_i$ is estimable, since it can be written as $\mathbf{a}'_i\mathbf{g}$, where $\mathbf{a}'_i$ is a row of $\mathbf{X}$, $i = 1, 2, \dots, a$. It follows that $\alpha_i - \alpha_{i'} = (\mathbf{a}'_i - \mathbf{a}'_{i'})\mathbf{g}$ is estimable, since the vector $\mathbf{a}'_i - \mathbf{a}'_{i'}$ belongs to the row space of $\mathbf{X}$.

- (b)** Suppose that μ is estimable; then it can be written as

$$\mu = \sum_{i=1}^a \tau_i (\mu + \alpha_i),$$

where $\tau_1, \tau_2, \dots, \tau_a$ are constants, since $\mu + \alpha_1, \mu + \alpha_2, \dots, \mu + \alpha_a$ form a basic set of estimable linear functions. We must therefore have $\sum_{i=1}^a \tau_i = 1$, and $\tau_i = 0$ for all i , a contradiction. Therefore, μ is nonestimable.

2.25. (a) $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ by Theorem 2.3.3(2).

(b) $E(\ell'\mathbf{y}) = \lambda'\boldsymbol{\beta} \Rightarrow \ell'\mathbf{X}\boldsymbol{\beta} = \lambda'\boldsymbol{\beta}$. Hence, $\lambda' = \ell'\mathbf{X}$. Now,

$$\begin{aligned}\text{Var}(\lambda'\hat{\boldsymbol{\beta}}) &= \lambda'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\lambda\sigma^2 \\ &= \ell'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\ell\sigma^2 \\ &= \ell'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\ell\sigma^2, \quad \text{by Theorem 2.3.3(2)} \\ &\leq \ell'\ell\sigma^2,\end{aligned}$$

since $\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ is positive semidefinite. The result follows from the last inequality, since $\text{Var}(\ell'\mathbf{y}) = \ell'\ell\sigma^2$,

2.26. $\boldsymbol{\beta}^* = \mathbf{P}(\Lambda + k\mathbf{I}_p)^{-1}\mathbf{P}'\mathbf{X}'\mathbf{y}$. But $\hat{\boldsymbol{\beta}} = \mathbf{P}\Lambda^{-1}\mathbf{P}'\mathbf{X}'\mathbf{y}$, so $\Lambda\mathbf{P}'\hat{\boldsymbol{\beta}} = \mathbf{P}'\mathbf{X}'\mathbf{y}$. Hence,

$$\begin{aligned}\boldsymbol{\beta}^* &= \mathbf{P}(\Lambda + k\mathbf{I}_p)^{-1}\Lambda\mathbf{P}'\hat{\boldsymbol{\beta}} \\ &= \mathbf{P}\mathbf{D}\mathbf{P}'\hat{\boldsymbol{\beta}}.\end{aligned}$$

2.27.

$$\begin{aligned}\sup_{\mathbf{x}, \mathbf{y}} \rho^2 &= \sup_{\boldsymbol{\nu}, \boldsymbol{\tau}} (\boldsymbol{\nu}'\mathbf{C}_1^{-1}\mathbf{A}\mathbf{C}_2^{-1}\boldsymbol{\tau})^2 \\ &= \sup_{\boldsymbol{\nu}} \left\{ \sup_{\boldsymbol{\tau}} (\boldsymbol{\nu}'\mathbf{C}_1^{-1}\mathbf{A}\mathbf{C}_2^{-1}\boldsymbol{\tau})^2 \right\} \\ &= \sup_{\boldsymbol{\nu}} \left\{ \sup_{\boldsymbol{\tau}} (\mathbf{b}'\boldsymbol{\tau})^2 \right\} \quad (\mathbf{b}' = \boldsymbol{\nu}'\mathbf{C}_1^{-1}\mathbf{A}\mathbf{C}_2^{-1}) \\ &= \sup_{\boldsymbol{\nu}} \left\{ \sup_{\boldsymbol{\tau}} (\boldsymbol{\tau}'\mathbf{b}\mathbf{b}'\boldsymbol{\tau}) \right\} \\ &= \sup_{\boldsymbol{\nu}} \{e_{\max}(\mathbf{b}\mathbf{b}')\}, \quad \text{by applying (2.9)} \\ &= \sup_{\boldsymbol{\nu}} \{e_{\max}(\mathbf{b}'\mathbf{b})\} \\ &= \sup_{\boldsymbol{\nu}} \{\boldsymbol{\nu}'\mathbf{C}_1^{-1}\mathbf{A}\mathbf{C}_2^{-2}\mathbf{A}'\mathbf{C}_1^{-1}\boldsymbol{\nu}\} \\ &= e_{\max}(\mathbf{C}_1^{-1}\mathbf{A}\mathbf{C}_2^{-2}\mathbf{A}'\mathbf{C}_1^{-1}), \quad \text{by (2.9)} \\ &= e_{\max}(\mathbf{C}_1^{-2}\mathbf{A}\mathbf{C}_2^{-2}\mathbf{A}') \\ &= e_{\max}(\mathbf{B}_1^{-1}\mathbf{A}\mathbf{B}_2^{-1}\mathbf{A}').\end{aligned}$$

CHAPTER 3

3.1. (a)

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1} &= \lim_{x \rightarrow 1} (1 + x + x^2 + x^3 + x^4) \\ &= 5.\end{aligned}$$

(b) $|x \sin(1/x)| \leq |x| \Rightarrow \lim_{x \rightarrow 0} x \sin(1/x) = 0.$

(c) The limit does not exist, since the function is equal to 1 except when $\sin(1/x) = 0$, that is, when $x = \mp 1/\pi, \mp 1/2\pi, \dots, \mp 1/n\pi, \dots$. For such values, the function takes the form $0/0$, and is therefore undefined. To have a limit at $x = 0$, the function must be defined at all points in a deleted neighborhood of $x = 0$.

(d) $\lim_{x \rightarrow 0^-} f(x) = \frac{1}{2}$ and $\lim_{x \rightarrow 0^+} f(x) = 1$. Therefore, the function $f(x)$ does not have a limit as $x \rightarrow 0$.

3.2. (a) To show that $(\tan x^3)/x^2 \rightarrow 0$ as $x \rightarrow 0$: $(\tan x^3)/x^2 = (1/\cos x^3)(\sin x^3)/x^2$. But $|\sin x^3| \leq |x^3|$. Hence, $|(\tan x^3)/x^2| \leq |x|/|\cos x^3| \rightarrow 0$ as $x \rightarrow 0$.

(b) $x/\sqrt{x} \rightarrow 0$ as $x \rightarrow 0$.

(c) $O(1)$ is bounded as $x \rightarrow \infty$. Therefore, $O(1)/x \rightarrow 0$ as $x \rightarrow \infty$, so $O(1) = o(x)$ as $x \rightarrow \infty$.

(d)

$$\begin{aligned}f(x)g(x) &= [x + o(x^2)] \left[\frac{1}{x^2} + O\left(\frac{1}{x}\right) \right] \\ &= \frac{1}{x} + x O\left(\frac{1}{x}\right) + \frac{1}{x^2} o(x^2) + o(x^2) O\left(\frac{1}{x}\right).\end{aligned}$$

Now, by definition, $|xO(1/x)|$ is bounded by a positive constant as $x \rightarrow 0$, and $o(x^2)/x^2 \rightarrow 0$ as $x \rightarrow 0$. Hence, $o(x^2)/x^2$ is bounded, that is, $O(1)$. Furthermore, $o(x^2)O(1/x) = x[o(x^2)/x^2]xO(1/x)$, which goes to zero as $x \rightarrow 0$, is also bounded. It follows that

$$f(x)g(x) = \frac{1}{x} + O(1).$$

3.3. (a) $f(0) = 0$ and $\lim_{x \rightarrow 0} f(x) = 0$. Hence, $f(x)$ is continuous at $x = 0$. It is also continuous at all other values of x .

(b) The function $f(x)$ is defined on $[1, 2]$, but $\lim_{x \rightarrow 2^-} f(x) = \infty$, so $f(x)$ is not continuous at $x = 2$.

- (c) If n is odd, then $f(x)$ is continuous everywhere, except at $x = 0$. If n is even, $f(x)$ will be continuous only when $x > 0$. (m and n must be expressed in their lowest terms so that the only common divisor is 1).
- (d) $f(x)$ is continuous for $x \neq 1$.

3.6.

$$f(x) = \begin{cases} -3, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Thus $f(x)$ is continuous everywhere except at $x = 0$.

3.7. $\lim_{x \rightarrow 0^-} f(x) = 2$ and $\lim_{x \rightarrow 0^+} f(x) = 0$. Therefore, $f(x)$ cannot be made continuous at $x = 0$.

3.8. Letting $a = b = 0$ in $f(a + b) = f(a) + f(b)$, we conclude that $f(0) = 0$. Now, for any $x_1, x_2 \in R$,

$$f(x_1) = f(x_2) + f(x_1 - x_2).$$

Let $z = x_1 - x_2$. Then

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(z)| \\ &= |f(z) - f(0)|. \end{aligned}$$

Since $f(x)$ is continuous at $x = 0$, for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(0)| < \epsilon$ if $|z| < \delta$. Hence, $|f(x_1) - f(x_2)| < \epsilon$ for all x_1, x_2 such that $|x_1 - x_2| < \delta$. Thus $f(x)$ is uniformly continuous everywhere in R .

3.9. $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1) = 1$, so $f(x)$ is continuous at $x = 1$. It is also continuous everywhere else on $[0, 2]$, which is closed and bounded. Therefore, it is uniformly continuous by Theorem 3.4.6.

3.10.

$$\begin{aligned} |\cos x_1 - \cos x_2| &= \left| 2 \sin\left(\frac{x_2 - x_1}{2}\right) \sin\left(\frac{x_2 + x_1}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x_2 - x_1}{2}\right) \right| \\ &\leq |x_1 - x_2| \quad \text{for all } x_1, x_2 \text{ in } R. \end{aligned}$$

Thus, for a given $\epsilon > 0$, there exists a $\delta > 0$ (namely, $\delta < \epsilon$) such that $|\cos x_1 - \cos x_2| < \epsilon$ whenever $|x_1 - x_2| < \delta$.

3.13. $f(x) = 0$ if x is a rational number in $[a, b]$. If x is an irrational number, then it is a limit of a sequence $\{y_n\}_{n=1}^{\infty}$ of rational numbers (any neighborhood of x contains infinitely many rationals). Hence,

$$f(x) = f\left(\lim_{n \rightarrow \infty} y_n\right) = \lim_{n \rightarrow \infty} f(y_n) = 0,$$

since $f(x)$ is a continuous function. Thus $f(x) = 0$ for every x in $[a, b]$.

3.14. $f(x)$ can be written as

$$f(x) = \begin{cases} 3 - 2x, & x \leq -1, \\ 5, & -1 < x < 1, \\ 3 + 2x, & x \geq 1. \end{cases}$$

Hence, $f(x)$ has a unique inverse for $x \leq -1$ and for $x \geq 1$.

3.15. The inverse function is

$$x = f^{-1}(y) = \begin{cases} y, & y \leq 1, \\ \frac{1}{2}(y + 1), & y > 1. \end{cases}$$

3.16. (a) The inverse of $f(x)$ is $f^{-1}(y) = 2 - \sqrt{y/2}$.

(b) The inverse of $f(x)$ is $f^{-1}(y) = 2 + \sqrt{y/2}$.

3.17. By Theorem 3.6.1,

$$\begin{aligned} \sum_{i=1}^n |f(a_i) - f(b_i)| &\leq K \sum_{i=1}^n |a_i - b_i| \\ &< K\delta. \end{aligned}$$

Choosing δ such that $\delta < \epsilon/K$, we get

$$\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon,$$

for any given $\epsilon > 0$.

3.18. This inequality can be proved by using mathematical induction: it is obviously true for $n = 1$. Suppose that it is true for $n = m$. To show that it is true for $n = m + 1$. For $n = m$, we have

$$f\left(\frac{\sum_{i=1}^m a_i x_i}{A_m}\right) \leq \frac{\sum_{i=1}^m a_i f(x_i)}{A_m},$$

where $A_m = \sum_{i=1}^m a_i$. Let

$$b_m = \frac{\sum_{i=1}^m a_i x_i}{A_m}.$$

Then

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{m+1} a_i x_i}{A_{m+1}}\right) &= f\left(\frac{A_m b_m + a_{m+1} x_{m+1}}{A_{m+1}}\right) \\ &\leq \frac{A_m}{A_{m+1}} f(b_m) + \frac{a_{m+1}}{A_{m+1}} f(x_{m+1}). \end{aligned}$$

But $f(b_m) \leq (1/A_m) \sum_{i=1}^m a_i f(x_i)$. Hence,

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{m+1} a_i x_i}{A_{m+1}}\right) &\leq \frac{\sum_{i=1}^m a_i f(x_i) + a_{m+1} f(x_{m+1})}{A_{m+1}} \\ &= \frac{\sum_{i=1}^{m+1} a_i f(x_i)}{A_{m+1}}. \end{aligned}$$

3.19. Let a be a limit point of S . There exists a sequence $\{a_n\}_{n=1}^\infty$ in S such that $\lim_{n \rightarrow \infty} a_n = a$ (if S is finite, then S is closed already). Hence, $f(a) = f(\lim_{n \rightarrow \infty} a_n) = \lim_{n \rightarrow \infty} f(a_n) = 0$. It follows that $a \in S$, and S is therefore a closed set.

3.20. Let $g(x) = \exp[f(x)]$. We have that

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for all x_1, x_2 in D , $0 \leq \lambda \leq 1$. Hence,

$$\begin{aligned} g[\lambda x_1 + (1 - \lambda)x_2] &\leq \exp[\lambda f(x_1) + (1 - \lambda)f(x_2)] \\ &\leq \lambda g(x_1) + (1 - \lambda)g(x_2), \end{aligned}$$

since the function e^x is convex. Hence, $g(x)$ is convex on D .

3.22. (a) We have that $E(|X|) \geq |E(X)|$. Let $X \sim N(\mu, \sigma^2)$. Then

$$E(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right] x dx.$$

Therefore,

$$\begin{aligned} |E(X)| &\leq \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] |x| \, dx \\ &= E(|X|). \end{aligned}$$

- (b) We have that $E(e^{-X}) \geq e^{-\mu}$, where $\mu = E(X)$, since e^{-x} is a convex function. The density function of the exponential distribution with mean μ is

$$g(x) = \frac{1}{\mu} e^{-x/\mu}, \quad 0 < x < \infty.$$

Hence,

$$\begin{aligned} E(e^{-X}) &= \frac{1}{\mu} \int_0^{\infty} e^{-x/\mu} e^{-x} \, dx \\ &= \frac{1}{\mu} \frac{1}{1 + 1/\mu} = \frac{1}{\mu + 1}. \end{aligned}$$

But

$$\begin{aligned} e^{\mu} &= 1 + \mu + \frac{1}{2!} \mu^2 + \cdots + \frac{1}{n!} \mu^n + \cdots \\ &\geq 1 + \mu. \end{aligned}$$

It follows that

$$e^{-\mu} \leq \frac{1}{\mu + 1}$$

and thus

$$E(e^{-X}) \geq e^{-\mu}.$$

3.23. $P(|X_n^2| \geq \epsilon) = P(|X_n| \geq \epsilon^{1/2}) \rightarrow 0$ as $n \rightarrow \infty$, since X_n converges in probability to zero.

3.24.

$$\begin{aligned} \sigma^2 &= E[(X - \mu)^2] \\ &= E[|X - \mu|^2] \\ &\geq E^2[|X - \mu|], \end{aligned}$$

hence,

$$\sigma \geq E[|X - \mu|].$$

CHAPTER 4

4.1.

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(h) - f(-h)}{2h} &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{2h} + \lim_{h \rightarrow 0} \frac{f(0) - f(-h)}{2h} \\
 &= \frac{f'(0)}{2} + \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{(-h)} \\
 &= \frac{f'(0)}{2} + \frac{f'(0)}{2} = f'(0).
 \end{aligned}$$

The converse is not true: let $f(x) = |x|$. Then

$$\frac{f(h) - f(-h)}{2h} = 0,$$

and its limit is zero. But $f'(0)$ does not exist.

4.3. For a given $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x \in N_\delta(x_0)$, $x \neq x_0$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon.$$

Hence,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| < |f'(x_0)| + \epsilon$$

and thus

$$|f(x) - f(x_0)| < A|x - x_0|,$$

where $A = |f'(x_0)| + \epsilon$.

4.4. $g(x) = f(x+1) - f(x) = f'(\xi)$, where ξ is between x and $x+1$. As $x \rightarrow \infty$, we have $\xi \rightarrow \infty$ and $f'(\xi) \rightarrow 0$, hence, $g(x) \rightarrow 0$ as $x \rightarrow \infty$.

4.5. We have that $f'(1) = \lim_{x \rightarrow 1^+} (x^3 - 2x + 1)/(x - 1) = \lim_{x \rightarrow 1^-} (ax^2 - bx + 1 + 1)/(x - 1)$, and thus $1 = 2a - b$. Furthermore, since $f(x)$ is continuous at $x = 1$, we must have $a - b + 1 = -1$. It follows that $a = 3$, $b = 5$, and

$$f'(x) = \begin{cases} 3x^2 - 2, & x \geq 1, \\ 6x - 5, & x < 1, \end{cases}$$

which is continuous everywhere.

4.6. Let $x > 0$.

(a) Taylor's theorem gives

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(\xi),$$

where $x < \xi < x + 2h$, and $h > 0$. Hence,

$$f'(x) = \frac{1}{2h} [f(x+2h) - f(x)] - hf''(\xi),$$

so that $|f'(x)| \leq m_0/h + hm_2$.

(b) Since m_1 is the least upper bound of $|f'(x)|$,

$$m_1 \leq \frac{m_0}{h} + hm_2.$$

Equivalently,

$$m_2h^2 - m_1h + m_0 \geq 0$$

This inequality is valid for all $h > 0$ provided that the discriminant

$$\Delta = m_1^2 - 4m_0m_2$$

is less than or equal to zero, that is, $m_1^2 \leq 4m_0m_2$.

4.7. No, unless $f'(x)$ is continuous at x_0 . For example, the function

$$f(x) = \begin{cases} x+1, & x > 0, \\ 0, & x = 0, \\ x-1, & x < 0, \end{cases}$$

does not have a derivative at $x = 0$, but $\lim_{x \rightarrow 0} f'(x) = 1$.

4.8.

$$\begin{aligned} D'(y_j) &= \lim_{a \rightarrow y_j} \frac{|y_j - a| + \sum_{i \neq j} |y_i - a| - \sum_{i=1}^n |y_i - y_j|}{a - y_j} \\ &= \lim_{a \rightarrow y_j} \frac{|y_j - a| + \sum_{i \neq j} |y_i - a| - \sum_{i \neq j} |y_i - y_j|}{a - y_j}, \end{aligned}$$

which does not exist, since $\lim_{a \rightarrow y_j} |y_j - a|/(a - y_j)$ does not exist.

4.9.

$$\frac{d}{dx} \left[\frac{f(x)}{x} \right] = \frac{xf'(x) - f(x)}{x^2}.$$

$xf'(x) - f(x) \rightarrow 0$ as $x \rightarrow 0$. Hence, by l'Hospital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{d}{dx} \left[\frac{f(x)}{x} \right] &= \lim_{x \rightarrow 0} \frac{f'(x) + xf''(x) - f'(x)}{2x} \\ &= \frac{1}{2} f''(0). \end{aligned}$$

4.10. Let $x_1, x_2 \in R$. Then,

$$|f(x_1) - f(x_2)| = |(x_1 - x_2)f'(\xi)|,$$

here ξ is between x_1 and x_2 . Since $f'(x)$ is bounded for all x , $|f'(\xi)| < M$ for some positive constant M . Hence, for a given $\epsilon > 0$, there exists a $\delta > 0$, where $M\delta < \epsilon$, such that $|f(x_1) - f(x_2)| < \epsilon$ if $|x_1 - x_2| < \delta$. Thus $f(x)$ is uniformly continuous on R .

4.11. $f'(x) = 1 + cg'(x)$, and $|cg'(x)| < cM$. Choose c such that $cM < \frac{1}{2}$. In this case,

$$|cg'(x)| < \frac{1}{2},$$

so

$$-\frac{1}{2} < cg'(x) < \frac{1}{2},$$

and,

$$\frac{1}{2} < f'(x) < \frac{3}{2}.$$

Hence, $f'(x)$ is positive and $f(x)$ is therefore strictly monotone increasing, thus $f(x)$ is one-to-one.

4.12. It is sufficient to show that $g'(x) \geq 0$ on $(0, \infty)$:

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}, \quad x > 0.$$

By the mean value theorem,

$$\begin{aligned} f(x) &= f(0) + xf'(c), \quad 0 < c < x \\ &= xf'(c). \end{aligned}$$

Hence,

$$g'(x) = \frac{xf'(x) - xf'(c)}{x^2} = \frac{f'(x) - f'(c)}{x}, \quad x > 0.$$

Since $f'(x)$ is monotone increasing, we have for all $x > 0$,

$$f'(x) \geq f'(c) \quad \text{and thus} \quad g'(x) \geq 0$$

4.14. Let $y = (1 + 1/x)^x$. Then,

$$\log y = x \log \left(1 + \frac{1}{x} \right) = \frac{\log(1 + 1/x)}{1/x}.$$

Applying l'Hospital's rule, we get

$$\lim_{x \rightarrow \infty} \log y = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2} \left(1 + \frac{1}{x} \right)^{-1}}{-\frac{1}{x^2}} = 1.$$

Therefore, $\lim_{x \rightarrow \infty} y = e$.

4.15. (a) $\lim_{x \rightarrow 0^+} f(x) = 1$, where $f(x) = (\sin x)^x$.

(b) $\lim_{x \rightarrow 0^+} g(x) = 0$, where $g(x) = e^{-1/x}/x$.

4.16. Let $f(x) = [1 + ax + o(x)]^{1/x}$, and let $y = ax + o(x)$. Then $y = x[a + o(1)]$, and $[1 + ax + o(x)]^{1/x} = (1 + y)^{a/y}(1 + y)^{o(1)/y}$. Now, as $x \rightarrow 0$, we have $y \rightarrow 0$, $(1 + y)^{a/y} \rightarrow e^a$, and $(1 + y)^{o(1)/y} \rightarrow 1$, since

$$\frac{o(1)}{y} \log(1 + y) \rightarrow 0 \quad \text{as } y \rightarrow 0.$$

It follows that $f(x) \rightarrow e^a$.

4.17. No, because both $f'(x)$ and $g'(x)$ vanish at $x = 0.541$ in $(0, 1)$.

4.18. Let $g(x) = f(x) - \gamma(x - a)$. Then

$$g'(x) = f'(x) - \gamma,$$

$$g'(a) = f'(a) - \gamma < 0,$$

$$g'(b) = f'(b) - \gamma > 0.$$

The function $g(x)$ is continuous on $[a, b]$. Therefore, it must achieve its absolute minimum at some point ξ in $[a, b]$. This point cannot be a or b , since $g'(a) < 0$ and $g'(b) > 0$. Hence, $a < \xi < b$, and $g'(\xi) = 0$, so $f'(\xi) = \gamma$.

4.19. Define $g(x) = f(x) - \tau(x - a)$, where

$$\tau = \sum_{i=1}^n \lambda_i f'(x_i).$$

There exist c_1, c_2 such that

$$\max_i f'(x_i) = f'(c_2), \quad a < c_2 < b,$$

$$\min_i f'(x_i) = f'(c_1), \quad a < c_1 < b.$$

If $f'(x_1) = f'(x_2) = \cdots = f'(x_n)$, then the result is obviously true. Let us therefore assume that these n derivatives are not all equal. In this case,

$$f'(c_1) < \tau < f'(c_2).$$

Apply now the result in Exercise 4.18 to conclude that there exists a point c between c_1 and c_2 such that $f'(c) = \tau$.

4.20. We have that

$$\sum_{i=1}^n [f(y_i) - f(x_i)] = \sum_{i=1}^n f'(c_i)(y_i - x_i),$$

where c_i is between x_i and y_i ($i = 1, 2, \dots, n$). Using Exercise 4.19, there exists a point c in (a, b) such that

$$\frac{\sum_{i=1}^n (y_i - x_i) f'(c_i)}{\sum_{i=1}^n (y_i - x_i)} = f'(c).$$

Hence,

$$\sum_{i=1}^n [f(y_i) - f(x_i)] = f'(c) \sum_{i=1}^n (y_i - x_i).$$

4.21. $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n / n, \quad |x| < 1.$

4.23. $f(x)$ has an absolute minimum at $x = \frac{1}{3}$.

- 4.24.** The function $f(x)$ is bounded if $x^2 + ax + b \neq 0$ for all x in $[-1, 1]$. Let $\Delta = a^2 - 4b$. If $\Delta < 0$, then $x^2 + ax + b > 0$ for all x . The denominator has an absolute minimum at $x = -a/2$. Thus, if $-1 \leq -a/2 \leq 1$, then $f(x)$ will have an absolute maximum at $x = -a/2$. Otherwise, $f(x)$ attains its absolute maximum at $x = -1$ or $x = 1$. If $\Delta = 0$, then

$$f(x) = \frac{1}{\left(x + \frac{a}{2}\right)^2}.$$

In this case, the point $-a/2$ must fall outside $[-1, 1]$, and the absolute maximum of $f(x)$ is attained at $x = -1$ or $x = 1$. Finally, if $\Delta > 0$, then

$$f(x) = \frac{1}{(x - x_1)(x - x_2)},$$

where $x_1 = \frac{1}{2}(-a - \sqrt{\Delta})$, $x_2 = \frac{1}{2}(-a + \sqrt{\Delta})$. Both x_1 and x_2 must fall outside $[-1, 1]$. In this case, the point $x = -a/2$ is equal to $\frac{1}{2}(x_1 + x_2)$, which falls outside $[-1, 1]$. Thus $f(x)$ attains its absolute maximum at $x = -1$ or $x = 1$.

- 4.25.** Let $H(y)$ denote the cumulative distribution function of $G^{-1}[F(Y)]$. Then

$$\begin{aligned} H(y) &= P\{G^{-1}[F(Y)] \leq y\} \\ &= P[F(Y) \leq G(y)] \\ &= P\{Y \leq F^{-1}[G(y)]\} \\ &= F\{F^{-1}[G(y)]\} \\ &= G(y). \end{aligned}$$

- 4.26.** Let $g(w)$ be the density function of W . Then $g(w) = 2we^{-w^2}$, $w \geq 0$.

- 4.27. (a)** Let $g(w)$ be the density function of W . Then

$$g(w) = \frac{1}{3w^{2/3}\sqrt{0.08\pi}} \exp\left[-\frac{1}{0.08}(w^{1/3} - 1)^2\right], \quad w \neq 0.$$

- (b) Exact mean is $E(W) = 1.12$. Exact variance is $\text{Var}(W) = 0.42$.
 (c) $E(w) \approx 1$, $\text{Var}(w) \approx 0.36$.

4.28. Let $G(y)$ be the cumulative distribution function of Y . Then

$$\begin{aligned}
 G(y) &= P(Y \leq y) \\
 &= P(Z^2 \leq y) \\
 &= P(|Z| \leq y^{1/2}) \\
 &= P(-y^{1/2} \leq Z \leq y^{1/2}) \\
 &= 2F(y^{1/2}) - 1,
 \end{aligned}$$

where $F(\cdot)$ is the cumulative distribution function of Z . Thus the density function of Y is $g(y) = 1/\sqrt{2\pi y} e^{-y/2}$, $y > 0$. This represents the density function of a chi-squared distribution with one degree of freedom.

4.29. (a) failure rate $= \frac{F(x+h) - F(x)}{1 - F(x)}.$

(b)

$$\begin{aligned}
 \text{Hazard rate} &= \frac{dF(x)/dx}{1 - F(x)} \\
 &= \frac{1}{e^{-x/\sigma}} \left(\frac{1}{\sigma} e^{-x/\sigma} \right) \\
 &= \frac{1}{\sigma}.
 \end{aligned}$$

(c) If

$$\frac{dF(x)/dx}{1 - F(x)} = c,$$

then

$$-\log[1 - F(x)] = cx + c_1,$$

hence,

$$1 - F(x) = c_2 e^{-cx}.$$

Since $F(0) = 0$, $c_2 = 1$. Therefore,

$$F(x) = 1 - e^{-cx}.$$

4.30.

$$\begin{aligned}
 P(Y_n = r) &= \frac{n(n-1) \cdots (n-r+1)}{r!} \left(\frac{\lambda t}{n} \right)^r \left(1 - \frac{\lambda t}{n} \right)^{n-r} \\
 &= \frac{n(n-1) \cdots (n-r+1)}{n^r} \frac{(\lambda t)^r}{r!} \left(1 - \frac{\lambda t}{n} \right)^{-r} \left(1 - \frac{\lambda t}{n} \right)^n.
 \end{aligned}$$

As $n \rightarrow \infty$, the first r factors on the right tend to 1, the next factor is fixed, the next tends to 1, and the last factor tends to $e^{-\lambda t}$. Hence,

$$\lim_{n \rightarrow \infty} P(Y_n = r) = \frac{e^{-\lambda t} (\lambda t)^r}{r!}.$$

CHAPTER 5

5.1. (a) The sequence $\{b_n\}_{n=1}^\infty$ is monotone increasing, since

$$\max\{a_1, a_2, \dots, a_n\} \leq \max\{a_1, a_2, \dots, a_{n+1}\}.$$

It is also bounded. Therefore, it is convergent by Theorem 5.1.2. Its limit is $\sup_{n \geq 1} a_n$, since $a_n \leq b_n \leq \sup_{n \geq 1} a_n$ for $n \geq 1$.

(b) Let $d_i = \log a_i - \log c$, $i = 1, 2, \dots$. Then

$$\log c_n = \frac{1}{n} \sum_{i=1}^n \log a_i = \log c + \frac{1}{n} \sum_{i=1}^n d_i.$$

To show that $(1/n) \sum_{i=1}^n d_i \rightarrow 0$ as $n \rightarrow \infty$: We have that $d_i \rightarrow 0$ as $i \rightarrow \infty$. Therefore, for a given $\epsilon > 0$, there exists a positive integer N_1 such that $|d_i| < \epsilon/2$ if $i > N_1$. Thus, for $n > N_1$,

$$\begin{aligned}
 \left| \frac{1}{n} \sum_{i=1}^n d_i \right| &\leq \frac{1}{n} \sum_{i=1}^{N_1} |d_i| + \frac{\epsilon}{2n} (n - N_1) \\
 &< \frac{1}{n} \sum_{i=1}^{N_1} |d_i| + \frac{\epsilon}{2}.
 \end{aligned}$$

Furthermore, there exists a positive integer N_2 such that

$$\frac{1}{n} \sum_{i=1}^{N_1} |d_i| < \frac{\epsilon}{2},$$

if $n > N_2$. Hence, if $n > \max(N_1, N_2)$, $|(1/n) \sum_{i=1}^n d_i| < \epsilon$, which implies that $(1/n) \sum_{i=1}^n d_i \rightarrow 0$, that is, $\log c_n \rightarrow \log c$, and $c_n \rightarrow c$ as $n \rightarrow \infty$.

- 5.2.** Let a and b be the limits of $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, respectively. These limits exist because the two sequences are of the Cauchy type. To show that $d_n \rightarrow d$, where $d = |a - b|$:

$$\begin{aligned} |d_n - d| &= ||a_n - b_n| - |a - b|| \\ &\leq |(a_n - b_n) - (a - b)| \\ &\leq |a_n - a| + |b_n - b|. \end{aligned}$$

It is now obvious that $d_n \rightarrow d$ as $a_n \rightarrow a$ and $b_n \rightarrow b$.

- 5.3.** Suppose that $a_n \rightarrow c$. Then, for a given $\epsilon > 0$, there exists a positive integer N such that $|a_n - c| < \epsilon$ if $n > N$. Let $b_n = a_{k_n}$ be the n th term of a subsequence. Since $k_n \geq n$, we have $|b_n - c| < \epsilon$ if $n > N$. Hence, $b_n \rightarrow c$. Vice versa, if every subsequence converges to c , then $a_n \rightarrow c$, since $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself.

- 5.4.** If E is not bounded, then there exists a subsequence $\{b_n\}_{n=1}^{\infty}$ such that $b_n \rightarrow \infty$, where $b_n = a_{k_n}$, $k_1 < k_2 < \dots < k_n < \dots$. This is a contradiction, since $\{a_n\}_{n=1}^{\infty}$ is bounded.

- 5.5. (a)** For a given $\epsilon > 0$, there exists a positive integer N such that $|a_n - c| < \epsilon$ if $n > N$. Thus, there is a positive constant M such that $|a_n - c| < M$ for all n . Now,

$$\begin{aligned} \left| \frac{\sum_{i=1}^n \alpha_i a_i}{\sum_{i=1}^n \alpha_i} - c \right| &= \frac{1}{\sum_{i=1}^n \alpha_i} \left| \sum_{i=1}^n \alpha_i (a_i - c) \right| \\ &= \frac{1}{\sum_{i=1}^n \alpha_i} \left| \sum_{i=1}^N \alpha_i (a_i - c) \right| + \frac{1}{\sum_{i=1}^n \alpha_i} \left| \sum_{i=N+1}^n \alpha_i (a_i - c) \right| \\ &\leq \frac{1}{\sum_{i=1}^n \alpha_i} \sum_{i=1}^N \alpha_i |a_i - c| + \frac{1}{\sum_{i=1}^n \alpha_i} \sum_{i=N+1}^n \alpha_i |a_i - c| \\ &\leq \frac{M}{\sum_{i=1}^n \alpha_i} \sum_{i=1}^N \alpha_i + \frac{\epsilon}{\sum_{i=1}^n \alpha_i} \sum_{i=N+1}^n \alpha_i \\ &< M \frac{\sum_{i=1}^N \alpha_i}{\sum_{i=1}^n \alpha_i} + \epsilon. \end{aligned}$$

Hence, as $n \rightarrow \infty$,

$$\left| \frac{\sum_{i=1}^n \alpha_i a_i}{\sum_{i=1}^n \alpha_i} - c \right| \rightarrow 0.$$

- (b) Let $a_n = (-1)^n$. Then, $\{a_n\}_{n=1}^\infty$ does not converge. But $(1/n)\sum_{i=1}^n a_i$ is equal to zero if n is even, and is equal to $-1/n$ if n is odd. Hence, $(1/n)\sum_{i=1}^n a_i$ goes to zero as $n \rightarrow \infty$.

5.6. For a given $\epsilon > 0$, there exists a positive integer N such that

$$\left| \frac{a_{n+1}}{a_n} - b \right| < \epsilon$$

if $n > N$. Since $b < 1$, we can choose ϵ so that $b + \epsilon < 1$. Then

$$\frac{a_{n+1}}{a_n} < b + \epsilon < 1, \quad n > N.$$

Hence, for $n \geq N + 1$,

$$\begin{aligned} a_{N+2} &< a_{N+1}(b + \epsilon), \\ a_{N+3} &< a_{N+2}(b + \epsilon) < a_{N+1}(b + \epsilon)^2, \\ &\vdots \\ a_n &< a_{N+1}(b + \epsilon)^{n-N-1} = \frac{a_{N+1}}{(b + \epsilon)^{N+1}}(b + \epsilon)^n. \end{aligned}$$

Letting $c = a_{N+1}/(b + \epsilon)^{N+1}$, $r = b + \epsilon$, we get $a_n < cr^n$, where $0 < r < 1$, for $n \geq N + 1$.

5.7. We first note that for each n , $a_n > 0$, which can be proved by induction. Let us consider two cases.

1. $b > 1$. In this case, we can show that the sequence is (i) bounded from above, and (ii) monotone increasing:
 - i. The sequence is bounded from above by $\sqrt{b}$, that is, $a_n < \sqrt{b}$: $a_n^2 < b$ is true for $n = 1$ because $a_1 = 1 < b$. Suppose now that $a_n^2 < b$; to show that $a_{n+1}^2 < b$:

$$\begin{aligned} b - a_{n+1}^2 &= b - \frac{a_n^2(3b + a_n^2)^2}{(3a_n^2 + b)^2} \\ &= \frac{(b - a_n^2)^3}{(3a_n^2 + b)^2} > 0. \end{aligned}$$

Thus $a_{n+1}^2 < b$, and the sequence is bounded from above by $\sqrt{b}$.

- ii. The sequence is monotone increasing: we have that $(3b + a_n^2)/(3a_n^2 + b) > 1$, since $a_n^2 < b$, as was seen earlier. Hence, $a_{n+1} > a_n$ for all n .

By Corollary 5.1.1(1), the sequence must be convergent. Let c be its limit. We then have the equation

$$c = \frac{c(3b + c^2)}{3c^2 + b},$$

which results from taking the limit of both sides of $a_{n+1} = a_n(3b + a_n^2)/(3a_n^2 + b)$ and noting that $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n = c$. The only solution to the above equation is $c = \sqrt{b}$.

2. $b < 1$. In this case, we can similarly show that the sequence is bounded from below and is monotone decreasing. Therefore, by Corollary 5.1.1(2) it must be convergent. Its limit is equal to $\sqrt{b}$.

- 5.8. The sequence is bounded from above by 3, that is, $a_n < 3$ for all n : This is true for $n = 1$. If it is true for n , then

$$a_{n+1} = (2 + a_n)^{1/2} < (2 + 3)^{1/2} < 3.$$

By induction, $a_n < 3$ for all n . Furthermore, the sequence is monotone increasing: $a_1 \leq a_2$, since $a_1 = 1$, $a_2 = \sqrt{3}$. If $a_n \leq a_{n+1}$, then

$$a_{n+2} = (2 + a_{n+1})^{1/2} \geq (2 + a_n)^{1/2} = a_{n+1}.$$

By induction, $a_n \leq a_{n+1}$ for all n . By Corollary 5.1.1(1) the sequence must be convergent. Let c be its limit, which can be obtained by solving the equation

$$c = (2 + c)^{1/2}.$$

The only solution is $c = 2$.

- 5.10. Let $m = 2n$. Then $a_m - a_n = 1/(n+1) + 1/(n+2) + \cdots + 1/2n$. Hence,

$$a_m - a_n > \frac{n}{2n} = \frac{1}{2}.$$

Therefore, $|a_m - a_n|$ cannot be made less than $\frac{1}{2}$ no matter how large m and n are, if $m = 2n$. This violates the condition for a sequence to be Cauchy.

5.11. For $m > n$, we have that

$$\begin{aligned}
 |a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) \\
 &\quad + \cdots + (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n)| \\
 &< br^{m-1} + br^{m-2} + \cdots + br^n \\
 &= br^n(1 + r + r^2 + \cdots + r^{m-n-1}) \\
 &= \frac{br^n(1 - r^{m-n})}{1 - r} < \frac{br^n}{1 - r}.
 \end{aligned}$$

For a given $\epsilon > 0$, we choose n large enough such that $br^n/(1 - r) < \epsilon$, which implies that $|a_m - a_n| < \epsilon$, that is, the sequence satisfies the Cauchy criterion; hence, it is convergent.

5.12. If $\{s_n\}_{n=1}^\infty$ is bounded from above, then it must be convergent, since it is monotone increasing and thus $\sum_{n=1}^\infty a_n$ converges. Vice versa, if $\sum_{n=1}^\infty a_n$ is convergent, then $\{s_n\}_{n=1}^\infty$ is a convergent sequence; hence, it is bounded, by Theorem 5.1.1.

5.13. Let s_n be the n th partial sum of the series. Then

$$\begin{aligned}
 s_n &= \frac{1}{3} \sum_{i=1}^n \left(\frac{1}{3i-1} - \frac{1}{3i+2} \right) \\
 &= \frac{1}{3} \left(\frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \cdots + \frac{1}{3n-1} - \frac{1}{3n+2} \right) \\
 &= \frac{1}{3} \left(\frac{1}{2} - \frac{1}{3n+2} \right) \\
 &\rightarrow \frac{1}{6} \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

5.14. For $n > 2$, the binomial theorem gives

$$\begin{aligned}
 \left(1 + \frac{1}{n}\right)^n &= 1 + \binom{n}{1} \frac{1}{n} + \binom{n}{2} \frac{1}{n^2} + \cdots + \frac{1}{n^n} \\
 &= 2 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n^n} \\
 &< 2 + \sum_{i=2}^n \frac{1}{i!} \\
 &< 2 + \sum_{i=2}^n \frac{1}{2^{i-1}} \\
 &< 3 \\
 &< n.
 \end{aligned}$$

Hence,

$$\left(1 + \frac{1}{n}\right) < n^{1/n},$$

$$(n^{1/n} - 1)^p > \frac{1}{n^p}.$$

Therefore, the series $\sum_{n=1}^{\infty} (n^{1/n} - 1)^p$ is divergent by the comparison test, since $\sum_{n=1}^{\infty} 1/n^p$ is divergent for $p \leq 1$.

5.15. (a) Suppose that $a_n < M$ for all n , where M is a positive constant. Then

$$\frac{a_n}{1 + a_n} > \frac{a_n}{1 + M}.$$

The series $\sum_{n=1}^{\infty} a_n/(1 + a_n)$ is divergent by the comparison test, since $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) We have that

$$\frac{a_n}{1 + a_n} = 1 - \frac{1}{1 + a_n}.$$

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then

$$\lim_{n \rightarrow \infty} a_n = \infty, \quad \text{hence,} \quad \lim_{n \rightarrow \infty} \frac{a_n}{1 + a_n} = 1 \neq 0.$$

Therefore, $\sum_{n=1}^{\infty} a_n/(1 + a_n)$ is divergent.

5.16. The sequence $\{s_n\}_{n=1}^{\infty}$ is monotone increasing. Hence, for $n = 2, 3, \dots$,

$$\begin{aligned} \frac{a_n}{s_n^2} &\leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}, \\ \sum_{i=1}^n \frac{a_i}{s_i^2} &= \frac{a_1}{s_1^2} + \sum_{i=2}^n \frac{a_i}{s_i^2} \\ &\leq \frac{a_1}{s_1^2} + \left(\frac{1}{s_1} - \frac{1}{s_2} \right) + \dots + \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) \\ &= \frac{a_1}{s_1^2} + \frac{1}{s_1} - \frac{1}{s_n} \rightarrow \frac{a_1}{s_1^2} + \frac{1}{s_1}, \end{aligned}$$

since $s_n \rightarrow \infty$ by the divergence of $\sum_{n=1}^{\infty} a_n$. It follows that $\sum_{n=1}^{\infty} a_n/s_n^2$ is a convergent series.

5.17. Let A denote the sum of the series $\sum_{n=1}^{\infty} a_n$. Then $r_n = A - s_{n-1}$, where $s_n = \sum_{i=1}^n a_i$. The sequence $\{r_n\}_{n=2}^{\infty}$ is monotone decreasing. Hence,

$$\begin{aligned} \sum_{i=m}^n \frac{a_i}{r_i} &> \frac{a_m + a_{m+1} + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m}. \end{aligned}$$

Since $r_n \rightarrow 0$, we have $1 - r_n/r_m \rightarrow 1$ as $n \rightarrow \infty$. Therefore, for $0 < \epsilon < 1$, there exists a positive integer k such that

$$\frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdots + \frac{a_{m+k}}{r_{m+k}} > \epsilon.$$

This implies that the series $\sum_{n=1}^{\infty} a_n/r_n$ does not satisfy the Cauchy criterion. Hence, it is divergent.

5.18. $(1/n)\log(1/n) = -(\log n)/n \rightarrow 0$ as $n \rightarrow \infty$. Hence, $(1/n)^{1/n} \rightarrow 1$. Similarly, $(1/n)\log(1/n^2) = -(2\log n)/n \rightarrow 0$, which implies $(1/n^2)^{1/n} \rightarrow 1$.

5.19. (a) $a_n^{1/n} = n^{1/n} - 1 \rightarrow 0 < 1$ as $n \rightarrow \infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ is convergent by the root test.

(b) $a_n < [\log(1+n)]/\log e^{n^2} = [\log(1+n)]/n^2$. Since

$$\begin{aligned} \int_1^{\infty} \frac{\log(1+x)}{x^2} dx &= -\frac{1}{x} \log(1+x) \Big|_1^{\infty} + \int_1^{\infty} \frac{dx}{x(x+1)} \\ &= \log 2 + \log \left(\frac{x}{x+1} \right) \Big|_1^{\infty} \\ &= 2 \log 2, \end{aligned}$$

the series $\sum_{n=1}^{\infty} [\log(1+n)]/n^2$ is convergent by the integral test. Hence, $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

(c) $a_n/a_{n+1} = (2n+2)(2n+3)/(2n+1)^2 \Rightarrow \lim_{n \rightarrow \infty} n(a_n/a_{n+1} - 1) = \frac{3}{2} > 1$. The series $\sum_{n=1}^{\infty} a_n$ is convergent by Raabe's test.

(d)

$$\begin{aligned} a_n &= \sqrt{n+2\sqrt{n}} - \sqrt{n} \\ &= \frac{n+2\sqrt{n}-n}{\sqrt{n+2\sqrt{n}}+\sqrt{n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent.

- (e) $|a_n|^{1/n} = 4/n \rightarrow 0 < 1$. The series is absolutely convergent by the root test.
- (f) $a_n = (-1)^n \sin(\pi/n)$. For large n , $\sin(\pi/n) \sim \pi/n$. Therefore, $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

5.20. (a) Applying the ratio test, we have the condition

$$|x|^2 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

which is equivalent to

$$\frac{|x|^2}{3} < 1, \quad \text{that is, } |x| < \sqrt{3}.$$

The series is divergent if $|x| = \sqrt{3}$. Hence, it is uniformly convergent on $[-r, r]$ where $r < \sqrt{3}$.

- (b) Using the ratio test, we get

$$|x| \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

where $a_n = 10^n/n$. This is equivalent to

$$10|x| < 1.$$

The series is divergent if $|x| = \frac{1}{10}$. Hence, it is uniformly convergent on $[-r, r]$ where $r < \frac{1}{10}$.

- (c) The series is uniformly convergent on $[-r, r]$ where $r < 1$.
- (d) The series is uniformly convergent everywhere by Weierstrass's M -test, since

$$\left| \frac{\cos nx}{n(n^2 + 1)} \right| \leq \frac{1}{n(n^2 + 1)} \quad \text{for all } x,$$

and the series whose n th term is $1/n(n^2 + 1)$ is convergent.

5.21. The series $\sum_{n=1}^{\infty} a_n$ is convergent by Theorem 5.2.14. Let s denote its sum:

$$\begin{aligned} s &= 1 - \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots \\ &< 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = \frac{10}{12}. \end{aligned}$$

Now,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n &= \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots \\ &\quad + \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right) + \cdots.\end{aligned}$$

Let s'_{3n} denote the sum of the first $3n$ terms of $\sum_{n=1}^{\infty} b_n$. Then,

$$s'_{3n} = s_{2n} + u_n,$$

where s_{2n} is the sum of the first $2n$ terms of $\sum_{n=1}^{\infty} a_n$, and

$$u_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1}, \quad n = 1, 2, \dots$$

The sequence $\{u_n\}_{n=1}^{\infty}$ is monotone increasing and is bounded from above by $\frac{1}{2}$, since

$$u_n < \frac{n}{2n+1}, \quad n = 1, 2, \dots$$

(the number of terms that make up u_n is equal to n). Thus $\{s'_{3n}\}_{n=1}^{\infty}$ is convergent, which implies convergence of $\sum_{n=1}^{\infty} b_n$. Let t denote the sum of this series. Note that

$$t > \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) = \frac{11}{12},$$

since $1/(4n-3) + 1/(4n-1) - 1/2n > 0$ for all n .

5.22. Let c_n denote the n th term of Cauchy's product. Then

$$\begin{aligned}c_n &= \sum_{k=0}^n a_k a_{n-k} \\ &= (-1)^n \sum_{k=0}^n \frac{1}{[(n-k+1)(k+1)]^{1/2}}.\end{aligned}$$

Since $n - k + 1 \leq n + 1$ and $k + 1 \leq n + 1$,

$$|c_n| \geq \sum_{k=0}^n \frac{1}{n+1} = 1.$$

Hence, c_n does not go to zero as $n \rightarrow \infty$. Therefore, $\sum_{n=0}^{\infty} c_n$ is divergent.

5.23. $f_n(x) \rightarrow f(x)$, where

$$f(x) = \begin{cases} 0, & x = 0, \\ 1/x, & x > 0. \end{cases}$$

The convergence is not uniform on $[0, \infty)$, since $f_n(x)$ is continuous on $[0, \infty)$ for all n , but $f(x)$ is discontinuous at $x = 0$.

5.24. (a) $1/n^x \leq 1/n^{1+\delta}$. Since $\sum_{n=1}^{\infty} 1/n^{1+\delta}$ is a convergent series, $\sum_{n=1}^{\infty} 1/n^x$ is uniformly convergent on $[1+\delta, \infty)$ by Weierstrass's M -test.

(b) $d/dx(1/n^x) = -(\log n)/n^x$. The series $\sum_{n=1}^{\infty} (\log n)/n^x$ is uniformly convergent on $[1+\delta, \infty)$:

$$\left| \frac{\log n}{n^x} \right| \leq \frac{1}{n^{1+\delta/2}} \frac{\log n}{n^{\delta/2}}, \quad x \geq \delta + 1.$$

Since $(\log n)/n^{\delta/2} \rightarrow 0$ as $n \rightarrow \infty$, there exists a positive integer N such that

$$\frac{\log n}{n^{\delta/2}} < 1 \quad \text{if } n > N.$$

Therefore,

$$\left| \frac{\log n}{n^x} \right| < \frac{1}{n^{1+\delta/2}}$$

if $n > N$ and $x \geq \delta + 1$. But, $\sum_{n=1}^{\infty} 1/n^{1+\delta/2}$ is convergent. Hence, $\sum_{n=1}^{\infty} (\log n)/n^x$ is uniformly convergent on $[1+\delta, \infty)$ by Weierstrass's M -test. If $\zeta(x)$ denotes the sum of the series $\sum_{n=1}^{\infty} 1/n^x$, then

$$\zeta'(x) = - \sum_{n=1}^{\infty} \frac{\log n}{n^x}, \quad x \geq \delta + 1.$$

5.25. We have that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n = \frac{1}{(1-x)^k}, \quad k = 1, 2, \dots,$$

if $-1 < x < 1$ (see Example 5.4.4). Differentiating this series term by term, we get

$$\sum_{n=1}^{\infty} nx^{n-1} \binom{n+k-1}{n} = \frac{k}{(1-x)^{k+1}}.$$

It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} n \binom{n+r-1}{r} p^r (1-p)^n &= p^r \frac{r(1-p)}{p^{r+1}} \\ &= \frac{r(1-p)}{p}. \end{aligned}$$

Taking now second derivatives, we obtain

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} \binom{n+k-1}{n} = \frac{k(k+1)}{(1-x)^{k+2}}.$$

From this we conclude that

$$\sum_{n=1}^{\infty} n^2 \binom{n+k-1}{n} x^n = \frac{kx(1+kx)}{(1-x)^{k+2}}.$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} n^2 \binom{n+r-1}{n} p^r (1-p)^n &= \frac{p^r(1-p)r[1+r(1-p)]}{p^{r+2}} \\ &= \frac{r(1-p)(1+r-rp)}{p^2}. \end{aligned}$$

5.26.

$$\begin{aligned} \phi(t) &= \sum_{n=0}^{\infty} e^{nt} \binom{n+r-1}{n} p^r q^n \\ &= p^r \sum_{n=0}^{\infty} \binom{n+r-1}{n} (qe^t)^n \\ &= \frac{p^r}{1-qe^t}, \quad qe^t < 1. \end{aligned}$$

The series converges uniformly on $(-\infty, s]$ where $s < -\log q$. Yes, formula (5.63) can be applied, since there exists a neighborhood $N_\delta(0)$ contained inside the interval of convergence by the fact that $-\log q > 0$.

5.28. It is sufficient to show that the series $\sum_{n=1}^{\infty} (\mu'_n/n!) \tau^n$ is absolutely convergent for some $\tau > 0$ (see Theorem 5.6.1). Using the root test,

$$\begin{aligned} \rho &= \limsup_{n \rightarrow \infty} \frac{|\mu'_n|^{1/n} \tau}{(n!)^{1/n}} \\ &= \tau e \limsup_{n \rightarrow \infty} \frac{|\mu'_n|^{1/n}}{(2\pi)^{1/2n} n^{1+1/2n}} \\ &= \tau e \limsup_{n \rightarrow \infty} \frac{|\mu'_n|^{1/n}}{n}. \end{aligned}$$

Let

$$m = \limsup_{n \rightarrow \infty} \frac{|\mu'_n|^{1/n}}{n}.$$

If $m > 0$ and finite, then the series $\sum_{n=1}^{\infty} (\mu'_n/n!) \tau^n$ is absolutely convergent if $\rho < 1$, that is, if $\tau < 1/(em)$.

5.30. (a)

$$\begin{aligned} E(X_n) &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} \\ &= p \sum_{k=1}^n k \binom{n}{k} p^{k-1} (1-p)^{n-k} \\ &= p \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} p^k (1-p)^{n-k-1} \\ &= p \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} p^k (1-p)^{n-k-1} \\ &= np(p+1-p)^{n-1} \\ &= np. \end{aligned}$$

We can similarly show that

$$E(X_n^2) = np(1-p) + n^2 p^2.$$

Hence,

$$\text{Var}(X_n) = E(X_n^2) - n^2 p^2$$

$$\text{Var}(X_n) = np(1-p).$$

- (b) We have that $P(|Y_n| \geq r\sigma) \leq 1/r^2$. Let $r\sigma = \epsilon$, where $\sigma^2 = p(1-p)/n$. Then

$$P(|Y_n| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}.$$

- (c) As $n \rightarrow \infty$, $P(|Y_n| \geq \epsilon) \rightarrow 0$, which implies that Y_n converges in probability to zero.

5.31. (a)

$$\begin{aligned}\phi_n(t) &= (p_n e^t + q_n)^n, \quad q_n = 1 - p_n \\ &= [1 + p_n(e^t - 1)]^n.\end{aligned}$$

Let $np_n = r_n$. Then

$$\begin{aligned}\phi_n(t) &= \left[1 + \frac{r_n}{n}(e^t - 1)\right]^n \\ &\rightarrow \exp[\mu(e^t - 1)]\end{aligned}$$

as $n \rightarrow \infty$. The limit of $\phi_n(t)$ is the moment generating function of a Poisson distribution with mean μ . Thus S_n has a limiting Poisson distribution with mean μ .

5.32. We have that

$$\begin{aligned}\mathbf{Q} &= (\mathbf{I}_n - \mathbf{H}_1 + k\mathbf{H}_2 - k^2\mathbf{H}_3 + k^3\mathbf{H}_4 - \cdots)^2 \\ &= (\mathbf{I}_n - \mathbf{H}_1) + k^2\mathbf{H}_3 - 2k^3\mathbf{H}_4 + \cdots.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{y}'\mathbf{Q}\mathbf{y} &= \mathbf{y}'(\mathbf{I}_n - \mathbf{H}_1)\mathbf{y} + k^2\mathbf{y}'\mathbf{H}_3\mathbf{y} - 2k^3\mathbf{y}'\mathbf{H}_4\mathbf{y} + \cdots \\ &= SS_E + k^2S_3 - 2k^3S_4 + \cdots \\ &= SS_E + \sum_{i=3}^{\infty} (i-2)(-k)^{i-1}S_i.\end{aligned}$$

CHAPTER 6

- 6.1.** If $f(x)$ is Riemann integrable, then inequality (6.1) is satisfied; hence, equality (6.6) is true, as was shown to be the case in the proof of Theorem 6.2.1. Vice versa, if equality (6.6) is satisfied, then by the double inequality (6.2), $S(P, f)$ must have a limit as $\Delta_p \rightarrow 0$, which implies that $f(x)$ is Riemann integrable on $[a, b]$.
- 6.2.** Consider the function $f(x)$ on $[0, 1]$ such that $f(x) = 0$ if $0 \leq x < \frac{1}{2}$, $f(x) = 2$ if $x = 1$, and $f(x) = \sum_{i=0}^n 1/2^i$ if $(n+1)/(n+2) \leq x < (n+2)/(n+3)$ for $n = 0, 1, 2, \dots$. This function has a countable number of discontinuities at $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$, but is Riemann integrable on $[0, 1]$, since it is monotone increasing (see Theorem 6.3.2).
- 6.3.** Suppose that $f(x)$ has one discontinuity of the first kind at $x = c$, $a < c < b$. Let $\lim_{x \rightarrow c^-} f(x) = L_1$, $\lim_{x \rightarrow c^+} f(x) = L_2$, $L_1 \neq L_2$. In any partition P of $[a, b]$, the point c appears in at most two subintervals. The contribution to $US_P(f) - LS_P(f)$ from these intervals is less than $2(M - m)\Delta_p$, where M, m are the supremum and infimum of $f(x)$ on $[a, b]$, and this can be made as small as we please. Furthermore, $\sum_i M_i \Delta x_i - \sum_i m_i \Delta x_i$ for the remaining subintervals can also be made as small as we please. Hence, $US_P(f) - LS_P(f)$ can be made smaller than ϵ for any given $\epsilon > 0$. Therefore, $f(x)$ is Riemann integrable on $[a, b]$. A similar argument can be used if $f(x)$ has a finite number of discontinuities of the first kind in $[a, b]$.
- 6.4.** Consider the following partition of $[0, 1]$: $P = \{0, 1/2n, 1/(2n-1), \dots, 1/3, 1/2, 1\}$. The number of partition points is $2n + 1$ including 0 and 1. In this case,

$$\begin{aligned}
 \sum_{i=1}^{2n} |\Delta f_i| &= \left| \frac{1}{2n} \cos \pi n - 0 \right| \\
 &+ \left| \frac{1}{2n-1} \cos \left[\frac{\pi}{2} (2n-1) \right] - \frac{1}{2n} \cos \pi n \right| \\
 &+ \left| \frac{1}{2n-2} \cos [\pi (n-1)] - \frac{1}{2n-1} \cos \left[\frac{\pi}{2} (2n-1) \right] \right| + \cdots \\
 &+ \left| \frac{1}{2} \cos \pi - \frac{1}{3} \cos \left(\frac{3\pi}{2} \right) \right| + \left| \cos \left(\frac{\pi}{2} \right) - \frac{1}{2} \cos \pi \right| \\
 &= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1.
 \end{aligned}$$

As $n \rightarrow \infty$, $\sum_{i=1}^{2n} |\Delta f_i| \rightarrow \infty$, since $\sum_{n=1}^{\infty} 1/n$ is divergent. Hence, $f(x)$ is not of bounded variation on $[0, 1]$.

6.5. (a) For a given $\epsilon > 0$, there exists a constant $M > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

if $x > M$. Since $g'(x) > 0$,

$$|f'(x) - Lg'(x)| < \epsilon g'(x).$$

Hence, if λ_1 and λ_2 are chosen larger than M , then

$$\begin{aligned} \left| \int_{\lambda_1}^{\lambda_2} [f'(x) - Lg'(x)] dx \right| &\leq \int_{\lambda_1}^{\lambda_2} |f'(x) - Lg'(x)| dx \\ &< \epsilon \int_{\lambda_1}^{\lambda_2} g'(x) dx. \end{aligned}$$

(b) From (a) we get

$$|f(\lambda_2) - Lg(\lambda_2) - f(\lambda_1) + Lg(\lambda_1)| < \epsilon [g(\lambda_2) - g(\lambda_1)].$$

Divide both sides by $g(\lambda_2)$ [which is positive for large λ_2 , since $g(x) \rightarrow \infty$ as $x \rightarrow \infty$], we obtain

$$\begin{aligned} \left| \frac{f(\lambda_2)}{g(\lambda_2)} - L - \frac{f(\lambda_1)}{g(\lambda_2)} + L \frac{g(\lambda_1)}{g(\lambda_2)} \right| &< \epsilon \left[1 - \frac{g(\lambda_1)}{g(\lambda_2)} \right] \\ &< \epsilon. \end{aligned}$$

Hence,

$$\left| \frac{f(\lambda_2)}{g(\lambda_2)} - L \right| < \epsilon + \frac{|f(\lambda_1)|}{g(\lambda_2)} + |L| \frac{g(\lambda_1)}{g(\lambda_2)}.$$

(c) For sufficiently large λ_2 , the second and third terms on the right-hand side of the above inequality can each be made smaller than ϵ ; hence,

$$\left| \frac{f(\lambda_2)}{g(\lambda_2)} - L \right| < 3\epsilon.$$

6.6. We have that

$$mg(x) \leq f(x)g(x) \leq Mg(x),$$

where m and M are the infimum and supremum of $f(x)$ on $[a, b]$, respectively. Let ξ and η be points in $[a, b]$ such that $m = f(\xi)$, $M = f(\eta)$. We conclude that

$$f(\xi) \leq \frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx} \leq f(\eta).$$

By the intermediate-value theorem (Theorem 3.4.4), there exists a constant c , between ξ and η , such that

$$\frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx} = f(c).$$

Note that c can be equal to ξ or η in case equality is attained at the lower end or the upper end of the above double inequality.

6.9. Integration by parts gives

$$\int_a^b f(x) dg(x) = f(b)g(b) - f(a)g(a) + \int_a^b g(x) d[-f(x)].$$

Since $g(x)$ is bounded, $f(b)g(b) \rightarrow 0$ as $b \rightarrow \infty$. Let us now establish convergence of $\int_a^b g(x) d[-f(x)]$ as $b \rightarrow \infty$: let $M > 0$ be such that $|g(x)| \leq M$ for all $x \geq a$. In addition,

$$\begin{aligned} \int_a^b M d[-f(x)] &= Mf(a) - Mf(b) \\ &\rightarrow Mf(a) \quad \text{as } b \rightarrow \infty. \end{aligned}$$

Hence, $\int_a^\infty M d[-f(x)]$ is a convergent integral. Since $-f(x)$ is monotone increasing on $[a, \infty)$, then

$$\int_a^\infty |g(x)| d[-f(x)] \leq \int_a^\infty M d[-f(x)].$$

This implies absolute convergence of $\int_a^\infty g(x) d[-f(x)]$. It follows that the integral $\int_a^\infty f(x) dg(x)$ is convergent.

6.10. Let n be a positive integer. Then

$$\begin{aligned}
 \int_0^{n\pi} \left| \frac{\sin x}{x} \right| dx &= \int_0^\pi \frac{\sin x}{x} dx - \int_\pi^{2\pi} \frac{\sin x}{x} dx \\
 &\quad + \cdots + (-1)^{n-1} \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx \\
 &= \int_0^\pi \sin x \left[\frac{1}{x} + \frac{1}{x+\pi} + \cdots + \frac{1}{x+(n-1)\pi} \right] dx \\
 &> \left(\frac{1}{\pi} + \frac{1}{2\pi} + \cdots + \frac{1}{n\pi} \right) \int_0^\pi \sin x dx \\
 &= \frac{2}{\pi} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \rightarrow \infty \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

6.11. (a) This is convergent by the integral test, since $\int_1^\infty (\log x)/(x\sqrt{x}) dx = 4 \int_0^\infty e^{-x} x dx = 4$ by a proper change of variable.

(b) $(n+4)/(2n^3+1) \sim 1/2n^2$. But $\sum_{n=1}^\infty 1/n^2$ is convergent by the integral test, since $\int_1^\infty dx/x^2 = 1$. Hence, this series is convergent by the comparison test.

(c) $1/(\sqrt{n+1}-1) \sim 1/\sqrt{n}$. By the integral test, $\sum_{n=1}^\infty 1/\sqrt{n}$ is divergent, since $\int_1^\infty dx/\sqrt{x} = 2\sqrt{x}|_1^\infty = \infty$. Hence, $\sum_{n=1}^\infty 1/(\sqrt{n+1}-1)$ is divergent.

6.13. (a) Near $x=0$ we have $x^{m-1}(1-x)^{n-1} \sim x^{m-1}$, and $\int_0^1 x^{m-1} dx = 1/m$. Also, near $x=1$ we have $x^{m-1}(1-x)^{n-1} \sim (1-x)^{n-1}$, and $\int_0^1 (1-x)^{n-1} dx = 1/n$. Hence, $B(m, n)$ converges if $m > 0$, $n > 0$.

(b) Let $\sqrt{x} = \sin \theta$. Then

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta.$$

(c) Let $x = 1/(1+y)$. Then

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy.$$

Letting $z = 1-x$, we get

$$\begin{aligned}
 \int_0^1 x^{m-1} (1-x)^{n-1} dx &= - \int_1^0 (1-z)^{m-1} z^{n-1} dz \\
 &= \int_0^1 x^{n-1} (1-x)^{m-1} dx.
 \end{aligned}$$

Hence, $B(m, n) = B(n, m)$. It follows that

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$$

(d)

$$\begin{aligned} B(m, n) &= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \\ &= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx. \end{aligned}$$

Let $y = 1/x$ in the second integral. We get

$$\int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Therefore,

$$\begin{aligned} B(m, n) &= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx \\ &= \int_0^1 \frac{x^{n-1} + x^{m-1}}{(1+x)^{m+n}} dx. \end{aligned}$$

6.14. (a)

$$\int_0^\infty \frac{dx}{\sqrt{1+x^3}} = \int_0^1 \frac{dx}{\sqrt{1+x^3}} + \int_1^\infty \frac{dx}{\sqrt{1+x^3}}.$$

The first integral exists because the integrand is continuous. The second integral is convergent because

$$\int_1^\infty \frac{dx}{\sqrt{1+x^3}} < \int_1^\infty \frac{dx}{x^{3/2}} = 2.$$

Hence, $\int_0^\infty dx/\sqrt{1+x^3}$ is convergent.

(b) Divergent, since for large x we have $1/(1+x^3)^{1/3} \sim 1/(1+x)$, and $\int_0^\infty dx/(1+x) = \infty$.

(c) Convergent, since if $f(x) = 1/(1-x^3)^{1/3}$ and $g(x) = 1/(1-x)^{1/3}$, then

$$\lim_{x \rightarrow 1^-} \frac{f(x)}{g(x)} = \left(\frac{1}{3}\right)^{1/3}.$$

But $\int_0^1 g(x) dx = \frac{3}{2}$. Hence, $\int_0^1 f(x) dx$ is convergent.

(d) Partition the integral as the sum of

$$\int_0^1 \frac{dx}{\sqrt{x}(1+x)} \quad \text{and} \quad \int_1^\infty \frac{dx}{\sqrt{x}(1+2x)}.$$

The first integral is convergent, since near $x=0$ we have $1/[\sqrt{x}(1+x)] \sim 1/\sqrt{x}$, and $\int_0^1 dx/\sqrt{x} = 2$. The second integral is also convergent, since as $x \rightarrow \infty$ we have $1/[\sqrt{x}(1+2x)] \sim 1/2x^{3/2}$, and $\int_1^\infty dx/x^{3/2} = 2$.

6.15. The proof of this result is similar to the proof of Corollary 6.4.2.

6.16. It is easy to show that

$$h'_n(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x).$$

Hence,

$$\begin{aligned} h_n(a) &= h_n(b) - \int_a^b h'_n(x) dx \\ &= \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx, \end{aligned}$$

since $h_n(b) = 0$. It follows that

$$\begin{aligned} f(b) &= f(a) + (b-a) f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \\ &\quad + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx. \end{aligned}$$

6.17. Let $G(x) = \int_a^x g(t) dt$. Then, $G(x)$ is uniformly continuous and $G'(x) = g(x)$ by Theorem 6.4.8. Thus

$$\begin{aligned} \int_a^b f(x) g(x) dx &= f(x) G(x) \Big|_a^b - \int_a^b G(x) f'(x) dx \\ &= f(b) G(b) - \int_a^b G(x) f'(x) dx. \end{aligned}$$

Now, since $G(x)$ is continuous, then by Corollary 3.4.1,

$$G(\xi) \leq G(x) \leq G(\eta)$$

for all x in $[a, b]$, where ξ, η are points in $[a, b]$ at which $G(x)$ achieves its infimum and supremum in $[a, b]$, respectively. Furthermore, since $f(x)$ is monotone (say monotone increasing) and $f'(x)$ exists, then $f'(x) \geq 0$ by Theorem 4.2.3. It follows that

$$G(\xi) \int_a^b f'(x) dx \leq \int_a^b G(x) f'(x) dx \leq G(\eta) \int_a^b f'(x) dx.$$

This implies that

$$\int_a^b G(x) f'(x) dx = \lambda \int_a^b f'(x) dx,$$

where $G(\xi) \leq \lambda \leq G(\eta)$. By Theorem 3.4.4, there exists a point c between ξ and η such that $\lambda = G(c)$. Hence,

$$\begin{aligned} \int_a^b G(x) f'(x) dx &= G(c) \int_a^b f'(x) dx \\ &= [f(b) - f(a)] \int_a^c g(x) dx. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_a^b f(x) g(x) dx &= f(b)G(b) - \int_a^b G(x) f'(x) dx \\ &= f(b) \int_a^b g(x) dx - [f(b) - f(a)] \int_a^c g(x) dx \\ &= f(a) \int_a^c g(x) dx + f(b) \int_c^b g(x) dx \end{aligned}$$

6.18. This follows directly from applying the result in Exercise 6.17 and letting $f(x) = 1/x$, $g(x) = \sin x$.

$$\begin{aligned} \int_a^b \frac{\sin x}{x} dx &= \frac{1}{a} \int_a^c \sin x dx + \frac{1}{b} \int_c^b \sin x dx \\ &= \frac{1}{a} (\cos a - \cos c) + \frac{1}{b} (\cos c - \cos b). \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_a^b \frac{\sin x}{x} dx \right| &\leq \frac{1}{a} |\cos a - \cos c| + \frac{1}{a} |\cos c - \cos b| \\ &\leq \frac{4}{a}. \end{aligned}$$

6.21. Let $\epsilon > 0$ be given. Choose $\eta > 0$ such that

$$\eta[g(b) - g(a)] < \epsilon.$$

Since $f(x)$ is uniformly continuous on $[a, b]$ (see Theorem 3.4.6), there exists a $\delta > 0$ such that

$$|f(x) - f(z)| < \eta$$

if $|x - z| < \delta$ for all x, z in $[a, b]$. Let us now choose a partition P of $[a, b]$ whose norm Δ_p is smaller than δ . Then,

$$\begin{aligned} US_P(f, g) - LS_P(f, g) &= \sum_{i=1}^n (M_i - m_i) \Delta g_i \\ &\leq \eta \sum_{i=1}^n \Delta g_i \\ &= \eta[g(b) - g(a)] \\ &< \epsilon. \end{aligned}$$

It follows that $f(x)$ is Riemann–Stieltjes integrable on $[a, b]$.

6.22. Since $g(x)$ is continuous, for any positive integer n we can choose a partition P such that

$$\Delta g_i = \frac{g(b) - g(a)}{n}, \quad i = 1, 2, \dots, n.$$

Also, since $f(x)$ is monotone increasing, then $M_i = f(x_i)$, $m_i = f(x_{i-1})$, $i = 1, 2, \dots, n$. Hence,

$$\begin{aligned} US_P(f, g) - LS_P(f, g) &= \frac{g(b) - g(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\ &= \frac{g(b) - g(a)}{n} [f(b) - f(a)]. \end{aligned}$$

This can be made less than ϵ , for any given $\epsilon > 0$, if n is chosen large enough. It follows that $f(x)$ is Riemann–Stieltjes integrable with respect to $g(x)$.

6.23. Let $f(x) = x^k/(1+x^2)$, $g(x) = x^{k-2}$. Then $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$. The integral $\int_0^\infty x^{k-2} dx$ is divergent, since $\int_0^\infty x^{k-2} dx = x^{k-1}/(k-1)|_0^\infty = \infty$ if $k > 1$. If $k = 1$, then $\int_0^\infty x^{k-2} dx = \int_0^\infty dx/x = \infty$. By Theorem 6.5.3, $\int_0^\infty f(x) dx$ must be divergent if $k \geq 1$. A similar procedure can be used to show that $\int_{-\infty}^0 f(x) dx$ is divergent.

6.24. Since $E(X) = 0$, then $\text{Var}(X) = E(X^2)$, which is given by

$$E(X^2) = \int_{-\infty}^{\infty} \frac{x^2 e^x}{(1 + e^x)^2} dx.$$

Let $u = e^x/(1 + e^x)$. The integral becomes

$$\begin{aligned} E(X^2) &= \int_0^1 \left[\log \left(\frac{u}{1-u} \right) \right]^2 du \\ &= \left\{ \log \left(\frac{u}{1-u} \right) [u \log u + (1-u) \log(1-u)] \right\}_0^1 \\ &\quad - \int_0^1 \frac{[u \log u + (1-u) \log(1-u)]}{u(1-u)} du \\ &= - \int_0^1 \frac{\log u}{1-u} du - \int_0^1 \frac{\log(1-u)}{u} du \\ &= -2 \int_0^1 \frac{\log u}{1-u} du. \end{aligned}$$

But

$$\int_0^1 \frac{\log u}{1-u} du = \int_0^1 (1 + u + u^2 + \cdots + u^n + \cdots) \log u du,$$

and

$$\begin{aligned} \int_0^1 u^n \log u du &= \frac{1}{n+1} u^{n+1} \log u \Big|_0^1 - \frac{1}{n+1} \int_0^1 u^n du \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\int_0^1 \frac{\log u}{1-u} du = - \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = -\frac{\pi^2}{6},$$

and $E(X^2) = \pi^2/3$.

6.26. Let $g(x) = f(x)/x^\alpha$, $h(x) = 1/x^\alpha$. We have that

$$\lim_{x \rightarrow 0^+} \frac{g(x)}{h(x)} = f(0) < \infty.$$

Since $\int_0^\delta h(x) dx = \delta^{1-\alpha}/(1-\alpha) < \infty$ for any $\delta > 0$, by Theorem 6.5.7, $\int_0^\delta f(x)/x^\alpha dx$ exists. Furthermore,

$$\int_\delta^\infty \frac{f(x)}{x^\alpha} dx \leq \frac{1}{\delta^\alpha} \int_\delta^\infty f(x) dx \leq \frac{1}{\delta^\alpha}.$$

Hence,

$$E(X^{-\alpha}) = \int_0^\infty \frac{f(x)}{x^\alpha} dx$$

exists.

6.27. Let $g(x) = f(x)/x^{1+\alpha}$, $h(x) = 1/x$. Then

$$\lim_{x \rightarrow 0^+} \frac{g(x)}{h(x)} = k < \infty.$$

Since $\int_0^\delta h(x) dx$ is divergent for any $\delta > 0$, then so is $\int_0^\delta g(x) dx$ by Theorem 6.5.7, and hence

$$E[X^{-(1+\alpha)}] = \int_0^\delta g(x) dx + \int_\delta^\infty g(x) dx = \infty.$$

6.28. Applying formula (6.84), the density function of W is given by

$$g(w) = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{w^2}{n}\right)^{-(n+1)/2}, \quad w \geq 0.$$

6.29. (a) From Theorem 6.9.2, we have

$$P(|X - \mu| \geq u\sigma) \leq \frac{1}{u^2}.$$

Letting $u\sigma = r$, we get

$$P(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2}.$$

(b) This follows from (a) and the fact that

$$E(\bar{X}_n) = \mu, \quad \text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

(c)

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Hence, as $n \rightarrow \infty$, $P(|\bar{X}_n - \mu| \geq \epsilon) \rightarrow 0$.

6.30. (a) Using the hint, for any u and $v > 0$ we have

$$u^2\nu_{k-1} + 2uv\nu_k + v^2\nu_{k+1} \geq 0.$$

Since $\nu_{k-1} \geq 0$ for $k \geq 1$, we must therefore have

$$v^2\nu_k^2 - v^2\nu_{k-1}\nu_{k+1} \leq 0, \quad k \geq 1,$$

that is,

$$\nu_k^2 \leq \nu_{k-1}\nu_{k+1}, \quad k = 1, 2, \dots, n-1.$$

(b) $\nu_0 = 1$. For $k = 1$, $\nu_1^2 \leq \nu_2$. Hence, $\nu_1 \leq \nu_2^{1/2}$. Let us now use mathematical induction to show that $\nu_n^{1/n} \leq \nu_{n+1}^{1/(n+1)}$ for all n for which ν_n and ν_{n+1} exist. The statement is true for $n = 1$. Suppose that

$$\nu_{n-1}^{1/(n-1)} \leq \nu_n^{1/n}.$$

To show that $\nu_n^{1/n} \leq \nu_{n+1}^{1/(n+1)}$: We have that

$$\begin{aligned} \nu_n^2 &\leq \nu_{n-1}\nu_{n+1} \\ &\leq \nu_n^{(n-1)/n}\nu_{n+1}. \end{aligned}$$

Thus,

$$\nu_n^{(n+1)/n} \leq \nu_{n+1}.$$

Hence,

$$\nu_n^{1/n} \leq \nu_{n+1}^{1/(n+1)}.$$

CHAPTER 7

7.1. (a) Along $x_1 = 0$, we have $f(0, x_2) = 0$ for all x_2 , and the limit is zero as $\mathbf{x} \rightarrow \mathbf{0}$. Any line through the origin can be represented by the equation $x_1 = tx_2$. Along this line, we have

$$\begin{aligned} f(tx_2, x_2) &= \frac{|tx_2|}{x_2^2} \exp\left(-\frac{|t||x_2|}{x_2^2}\right) \\ &= \frac{|t|}{|x_2|} \exp\left(-\frac{|t|}{|x_2|}\right), \quad x_2 \neq 0. \end{aligned}$$

Using l'Hospital's rule (Theorem 4.2.6), $f(tx_2, x_2) \rightarrow 0$ as $x_2 \rightarrow 0$.

(b) Along the parabola $x_1 = x_2^2$, we have

$$\begin{aligned} f(x_1, x_2) &= \frac{x_2^2}{x_2^2} \exp\left(-\frac{x_2^2}{x_2^2}\right) \\ &= e^{-1}, \quad x_2 \neq 0, \end{aligned}$$

which does not go to zero as $x_2 \rightarrow 0$. By Definition 7.2.1, $f(x_1, x_2)$ does not have a limit as $\mathbf{x} \rightarrow \mathbf{0}$.

7.5. (a) This function has no limit as $\mathbf{x} \rightarrow \mathbf{0}$, as was shown in Example 7.2.2. Hence, it is not continuous at the origin.

(b)

$$\begin{aligned} \left. \frac{\partial f(x_1, x_2)}{\partial x_1} \right|_{\mathbf{x}=\mathbf{0}} &= \lim_{\Delta x_1 \rightarrow 0} \left\{ \frac{1}{\Delta x_1} \left[\frac{0\Delta x_1}{(\Delta x_1)^2 + 0} - 0 \right] \right\} = 0, \\ \left. \frac{\partial f(x_1, x_2)}{\partial x_2} \right|_{\mathbf{x}=\mathbf{0}} &= \lim_{\Delta x_2 \rightarrow 0} \left\{ \frac{1}{\Delta x_2} \left[\frac{0\Delta x_2}{0 + (\Delta x_2)^2} - 0 \right] \right\} = 0. \end{aligned}$$

7.6. Applying formula (7.12), we get

$$\frac{df}{dt} = \sum_{i=1}^k x_i \frac{\partial f(\mathbf{u})}{\partial u_i} = nt^{n-1}f(\mathbf{x}),$$

where u_i is the i th element of $\mathbf{u} = t\mathbf{x}$, $i = 1, 2, \dots, k$. But, on one hand,

$$\begin{aligned} \frac{\partial f(\mathbf{u})}{\partial x_i} &= \sum_{j=1}^k \frac{\partial f(\mathbf{u})}{\partial u_j} \frac{\partial u_j}{\partial x_i} \\ &= t \frac{\partial f(\mathbf{u})}{\partial u_i}, \end{aligned}$$

and on the other hand,

$$\frac{\partial f(\mathbf{u})}{\partial x_i} = t^n \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

Hence,

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = t^{n-1} \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

It follows that

$$\sum_{i=1}^k x_i \frac{\partial f(\mathbf{u})}{\partial u_i} = nt^{n-1}f(\mathbf{x})$$

can be written as

$$t^{n-1} \sum_{i=1}^k x_i \frac{\partial f(\mathbf{x})}{\partial x_i} = nt^{n-1} f(\mathbf{x}),$$

that is,

$$\sum_{i=1}^k x_i \frac{\partial f(\mathbf{x})}{\partial x_i} = nf(\mathbf{x}).$$

- 7.7. (a)** $f(x_1, x_2)$ is not continuous at the origin, since along $x_1 = 0$ and $x_2 \neq 0$, $f(x_1, x_2) = 0$, which has a limit equal to zero as $x_2 \rightarrow 0$. But, along the parabola $x_2 = x_1^2$,

$$f(x_1, x_2) = \frac{x_1^4}{x_1^4 + x_1^4} = \frac{1}{2} \quad \text{if } x_1 \neq 0.$$

Hence, $\lim_{x_1 \rightarrow 0} f(x_1, x_1^2) = \frac{1}{2} \neq 0$.

- (b)** The directional derivative in the direction of the unit vector $\mathbf{v} = (v_1, v_2)'$ at the origin is given by

$$v_1 \frac{\partial f(\mathbf{x})}{\partial x_1} \Big|_{\mathbf{x}=\mathbf{0}} + v_2 \frac{\partial f(\mathbf{x})}{\partial x_2} \Big|_{\mathbf{x}=\mathbf{0}}.$$

This derivative is equal to zero, since

$$\begin{aligned} \frac{\partial f(\mathbf{x})}{\partial x_1} \Big|_{\mathbf{x}=\mathbf{0}} &= \lim_{\Delta x_1 \rightarrow 0} \frac{1}{\Delta x_1} \left[\frac{(\Delta x_1)^2 0}{(\Delta x_1)^4 + 0} - 0 \right] = 0, \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \Big|_{\mathbf{x}=\mathbf{0}} &= \lim_{\Delta x_2 \rightarrow 0} \frac{1}{\Delta x_2} \left[\frac{0 \Delta x_2}{0 + (\Delta x_2)^2} - 0 \right] = 0. \end{aligned}$$

- 7.8.** The directional derivative of f at a point $\mathbf{x}$ on C in the direction of $\mathbf{v}$ is $\sum_{i=1}^k v_i \partial f(\mathbf{x}) / \partial x_i$. But $\mathbf{v}$ is given by

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{x}}{dt} \Big/ \left\| \frac{d\mathbf{x}}{dt} \right\| \\ &= \frac{d\mathbf{x}}{dt} \Big/ \left\{ \sum_{i=1}^k \left[\frac{dg_i(t)}{dt} \right]^2 \right\}^{1/2} \\ &= \frac{d\mathbf{x}}{dt} \Big/ \left\| \frac{d\mathbf{g}}{dt} \right\|, \end{aligned}$$

where $\mathbf{g}$ is the vector whose i th element is $g_i(t)$. Hence,

$$\sum_{i=1}^k v_i \frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{1}{\|d\mathbf{g}/dt\|} \sum_{i=1}^k \frac{dg_i(t)}{dt} \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

Furthermore,

$$\frac{ds}{dt} = \left\| \frac{d\mathbf{g}}{dt} \right\|.$$

It follows that

$$\begin{aligned} \sum_{i=1}^k v_i \frac{\partial f(\mathbf{x})}{\partial x_i} &= \frac{dt}{ds} \sum_{i=1}^k \frac{dg_i(t)}{dt} \frac{\partial f(\mathbf{x})}{\partial x_i} \\ &= \sum_{i=1}^k \frac{dx_i}{ds} \frac{\partial f(\mathbf{x})}{\partial x_i} \\ &= \frac{df(\mathbf{x})}{ds}. \end{aligned}$$

7.9. (a)

$$\begin{aligned} f(x_1, x_2) &\approx f(0, 0) + \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f(0, 0) \\ &\quad + \frac{1}{2!} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} + 2x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + x_2^2 \frac{\partial^2}{\partial x_2^2} \right) f(0, 0) \\ &= 1 + x_1 x_2. \end{aligned}$$

(b)

$$\begin{aligned} f(x_1, x_2, x_3) &\approx f(0, 0, 0) + \left(\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} \right) f(0, 0, 0) \\ &\quad + \frac{1}{2!} \left(\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} \right)^2 f(0, 0, 0) \\ &= \sin(1) + x_1 \cos(1) + \frac{1}{2!} \{ x_1^2 [\cos(1) - \sin(1)] + 2x_2^2 \cos(1) \}. \end{aligned}$$

(c)

$$\begin{aligned} f(x_1, x_2) &\approx f(0, 0) + \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f(0, 0) \\ &\quad + \frac{1}{2!} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} + 2x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + x_2^2 \frac{\partial^2}{\partial x_2^2} \right) f(0, 0) \\ &= 1, \end{aligned}$$

since all first-order and second-order partial derivatives vanish at $\mathbf{x} = \mathbf{0}$.

- 7.10. (a)** If $u_1 = f(x_1, x_2)$ and $\partial f / \partial x_1 \neq 0$ at $\mathbf{x}_0$, then by the implicit function theorem (Theorem 7.6.2), there is neighborhood of $\mathbf{x}_0$ in which the equation $u_1 = f(x_1, x_2)$ can be solved uniquely for x_1 in terms of x_2 and u_1 , that is, $x_1 = h(u_1, x_2)$. Thus,

$$f[h(u_1, x_2), x_2] \equiv u_1.$$

Hence, by differentiating this identity with respect to x_2 , we get

$$\frac{\partial f}{\partial x_1} \frac{\partial h}{\partial x_2} + \frac{\partial f}{\partial x_2} = 0,$$

which gives

$$\frac{\partial h}{\partial x_2} = - \frac{\partial f}{\partial x_2} \bigg/ \frac{\partial f}{\partial x_1}$$

in a neighborhood of $\mathbf{x}_0$.

- (b)** On the basis of part (a), we can consider $g(x_1, x_2)$ as a function of x_2 and u , since $x_1 = h(u_1, x_2)$ in a neighborhood of $\mathbf{x}_0$. In such a neighborhood, the partial derivative of g with respect to x_2 is

$$\frac{\partial g}{\partial x_1} \frac{\partial h}{\partial x_2} + \frac{\partial g}{\partial x_2} = \frac{\partial(f, g)}{\partial(x_1, x_2)} \bigg/ \frac{\partial f}{\partial x_1},$$

which is equal to zero because $\partial(f, g)/\partial(x_1, x_2) = 0$. [Recall that $\partial h/\partial x_2 = -(\partial f/\partial x_2)(\partial h/\partial x_1)$.]

- (c)** From part (b), $g[h(u_1, x_2), x_2]$ is independent of x_2 in a neighborhood of $\mathbf{x}_0$. We can then write $\phi(u_1) = g[h(u_1, x_2), x_2]$. Since $x_1 = h(u_1, x_2)$ is equivalent to $u_1 = f(x_1, x_2)$ in a neighborhood of $\mathbf{x}_0$, $\phi[f(x_1, x_2)] = g(x_1, x_2)$ in this neighborhood.
- (d)** Let $G(f, g) = g(x_1, x_2) - \phi[f(x_1, x_2)]$. Then, in a neighborhood of $\mathbf{x}_0$, $G(f, g) = 0$. Hence,

$$\frac{\partial G}{\partial f} \frac{\partial f}{\partial x_1} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial x_1} \equiv 0,$$

$$\frac{\partial G}{\partial f} \frac{\partial f}{\partial x_2} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial x_2} \equiv 0.$$

In order for these two identities to be satisfied by values of $\partial G/\partial f$, $\partial G/\partial g$ not all zero, it is necessary that $\partial(f, g)/\partial(x_1, x_2)$ be equal to zero in this neighborhood of $\mathbf{x}_0$.

7.11. $u(x_1, x_2, x_3) = u(\xi_1 \xi_3, \xi_2 \xi_3, \xi_3)$. Hence,

$$\begin{aligned}\frac{\partial u}{\partial \xi_3} &= \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial \xi_3} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial \xi_3} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial \xi_3} \\ &= \xi_1 \frac{\partial u}{\partial x_1} + \xi_2 \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3},\end{aligned}$$

so that

$$\begin{aligned}\xi_3 \frac{\partial u}{\partial \xi_3} &= \xi_1 \xi_3 \frac{\partial u}{\partial x_1} + \xi_2 \xi_3 \frac{\partial u}{\partial x_2} + \xi_3 \frac{\partial u}{\partial x_3} \\ &= x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + x_3 \frac{\partial u}{\partial x_3} = nu.\end{aligned}$$

Integrating this partial differential equation with respect to ξ_3 , we get

$$\log u = n \log \xi_3 + \psi(\xi_1, \xi_2),$$

where $\psi(\xi_1, \xi_2)$ is a function of ξ_1, ξ_2 . Hence,

$$\begin{aligned}u &= \xi_3^n \exp[\psi(\xi_1, \xi_2)] \\ &= x_3^n F(\xi_1, \xi_2) \\ &= x_3^n F\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),\end{aligned}$$

where $F(\xi_1, \xi_2) = \exp[\psi(\xi_1, \xi_2)]$.

7.13. (a) The Jacobian determinant is $27x_1^2 x_2^2 x_3^2$, which is zero in any subset of R^3 that contains points on any of the coordinate planes.

(b) The unique inverse function is given by

$$x_1 = u_1^{1/3}, \quad x_2 = u_2^{1/3}, \quad x_3 = u_3^{1/3}.$$

7.14. Since $\partial(g_1, g_2)/\partial(x_1, x_2) \neq 0$, we can solve for x_1, x_2 uniquely in terms of y_1 and y_2 by Theorem 7.6.2. Differentiating g_1 and g_2 with respect to y_1 , we obtain

$$\begin{aligned}\frac{\partial g_1}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \frac{\partial g_1}{\partial x_2} \frac{\partial x_2}{\partial y_1} + \frac{\partial g_1}{\partial y_1} &= 0, \\ \frac{\partial g_2}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \frac{\partial g_2}{\partial x_2} \frac{\partial x_2}{\partial y_1} + \frac{\partial g_2}{\partial y_1} &= 0,\end{aligned}$$

Solving for $\partial x_1/\partial y_1$ and $\partial x_2/\partial y_1$ yields the desired result.

7.15. This is similar to Exercise 7.14. Since $\partial(f, g)/\partial(x_1, x_2) \neq 0$, we can solve for x_1, x_2 in terms of x_3 , and we get

$$\frac{dx_1}{dx_3} = - \frac{\partial(f, g)}{\partial(x_3, x_2)} \bigg/ \frac{\partial(f, g)}{\partial(x_1, x_2)},$$

$$\frac{dx_2}{dx_3} = - \frac{\partial(f, g)}{\partial(x_1, x_3)} \bigg/ \frac{\partial(f, g)}{\partial(x_1, x_2)}.$$

But

$$\frac{\partial(f, g)}{\partial(x_3, x_2)} = - \frac{\partial(f, g)}{\partial(x_2, x_3)}, \quad \frac{\partial(f, g)}{\partial(x_1, x_3)} = - \frac{\partial(f, g)}{\partial(x_3, x_1)}.$$

Hence,

$$\frac{dx_1}{\partial(f, g)/\partial(x_2, x_3)} = \frac{dx_3}{\partial(f, g)/\partial(x_1, x_2)} = \frac{dx_2}{\partial(f, g)/\partial(x_3, x_1)}.$$

7.16. (a) $(-\frac{1}{3}, -\frac{1}{3})$ is a point of local minimum.

(b)

$$\frac{\partial f}{\partial x_1} = 4\alpha x_1 - x_2 + 1 = 0,$$

$$\frac{\partial f}{\partial x_2} = -x_1 + 2x_2 - 1 = 0.$$

The solution of these two equations is $x_1 = 1/(1 - 8\alpha)$, $x_2 = (4\alpha - 1)/(8\alpha - 1)$. The Hessian matrix of f is

$$\mathbf{A} = \begin{bmatrix} 4\alpha & -1 \\ -1 & 2 \end{bmatrix}.$$

Here, $f_{11} = 4\alpha$, and $\det(\mathbf{A}) = 8\alpha - 1$.

i. Yes, if $\alpha > \frac{1}{8}$.

ii. No, it is not possible.

iii. Yes, if $\alpha < \frac{1}{8}$, $\alpha \neq 0$.

(c) The stationary points are $(-2, -2)$, $(4, 4)$. The first is a saddle point, and the second is a point of local minimum.

(d) The stationary points are $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, and $(0, 0)$. The first two are points of local minimum. At $(0, 0)$, $f_{11}f_{22} - f_{12}^2 = 0$. In

this case, $\mathbf{h}'\mathbf{A}\mathbf{h}$ has the same sign as $f_{11}|_{(0,0)} = -4$ for all values of $\mathbf{h} = (h_1, h_2)$, except when

$$h_1 + h_2 \frac{f_{12}}{f_{11}} = 0$$

or $h_1 - h_2 = 0$, in which case $\mathbf{h}'\mathbf{A}\mathbf{h} = 0$. For such values of $\mathbf{h}$, $(\mathbf{h}'\nabla)^3 f(0,0) = 0$, but $(\mathbf{h}'\nabla)^4 f(0,0) = 24(h_1^4 + h_2^4) = 48h_1^4$, which is nonnegative. Hence, the point $(0,0)$ is a saddlepoint.

7.19. The Lagrange equations are

$$(2 + 8\lambda)x_1 + 12x_2 = 0,$$

$$12x_1 + (4 + 2\lambda)x_2 = 0,$$

$$4x_1^2 + x_2^2 = 25.$$

We must have

$$(4\lambda + 1)(\lambda + 2) - 36 = 0.$$

The solutions are $\lambda_1 = -4.25$, $\lambda_2 = 2$. For $\lambda_1 = -4.25$, we have the points

i. $x_1 = 1.5$, $x_2 = 4.0$,

ii. $x_1 = -1.5$, $x_2 = -4.0$.

For $\lambda_2 = 2$, we have the points

iii. $x_1 = 2$, $x_2 = -3$,

iv. $x_1 = -2$, $x_2 = 3$.

The matrix $\mathbf{B}_1$ has the value

$$\mathbf{B}_1 = \begin{bmatrix} 2 + 8\lambda & 12 & 8x_1 \\ 12 & 4 + 2\lambda & 2x_2 \\ 8x_1 & 2x_2 & 0 \end{bmatrix}.$$

The determinant of $\mathbf{B}_1$ is Δ_1 .

At (i), $\Delta_1 = 5000$; the point is a local maximum.

At (ii), $\Delta_1 = 5000$; the point is a local maximum.

At (iii) and (iv), $\Delta_1 = -5000$; the points are local minima.

7.21. Let $F = x_1^2 x_2^2 x_3^2 + \lambda(x_1^2 + x_2^2 + x_3^2 - c^2)$. Then

$$\frac{\partial F}{\partial x_1} = 2x_1 x_2^2 x_3^2 + 2\lambda x_1 = 0,$$

$$\frac{\partial F}{\partial x_2} = 2x_1^2 x_2 x_3^2 + 2\lambda x_2 = 0,$$

$$\frac{\partial F}{\partial x_3} = 2x_1^2 x_2^2 x_3 + 2\lambda x_3 = 0,$$

$$x_1^2 + x_2^2 + x_3^2 = c^2.$$

Since x_1, x_2, x_3 cannot be equal to zero, we must have

$$x_2^2 x_3^2 + \lambda = 0,$$

$$x_1^2 x_3^2 + \lambda = 0,$$

$$x_1^2 x_2^2 + \lambda = 0.$$

These equations imply that $x_1^2 = x_2^2 = x_3^2 = c^2/3$ and $\lambda = -c^4/9$. For these values, it can be verified that $\Delta_1 < 0$, $\Delta_2 > 0$, where Δ_1 and Δ_2 are the determinants of $\mathbf{B}_1$ and $\mathbf{B}_2$, and

$$\mathbf{B}_1 = \begin{bmatrix} 2x_2^2 x_3^2 + 2\lambda & 4x_1 x_2 x_3^2 & 4x_1 x_2^2 x_3 & 2x_1 \\ 4x_1 x_2 x_3^2 & 2x_1^2 x_3^2 + 2\lambda & 4x_1^2 x_2 x_3 & 2x_2 \\ 4x_1 x_2^2 x_3 & 4x_1^2 x_2 x_3 & 2x_1^2 x_2^2 + 2\lambda & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 2x_1^2 x_3^2 + 2\lambda & 4x_1^2 x_2 x_3 & 2x_2 \\ 4x_1^2 x_2 x_3 & 2x_1^2 x_2^2 + 2\lambda & 2x_3 \\ 2x_2 & 2x_3 & 0 \end{bmatrix}.$$

Since $n = 3$ is odd, the function $x_1^2 x_2^2 x_3^2$ must attain a maximum value given by $(c^2/3)^3$. It follows that for all values of x_1, x_2, x_3 ,

$$x_1^2 x_2^2 x_3^2 \leq \left(\frac{c^2}{3}\right)^3,$$

that is,

$$(x_1^2 x_2^2 x_3^2)^{1/3} \leq \frac{x_1^2 + x_2^2 + x_3^2}{3}.$$

7.24. The domain of integration, D , is the region bounded from above by the parabola $x_2 = 4x_1 - x_1^2$ and from below by the parabola $x_2 = x_1^2$. The two parabolas intersect at $(0, 0)$ and $(2, 4)$. It is then easy to see that

$$\int_0^2 \left[\int_{x_1^2}^{4x_1 - x_1^2} f(x_1, x_2) dx_2 \right] dx_1 = \int_0^4 \left[\int_{2 - \sqrt{4 - x_2}}^{\sqrt{x_2}} f(x_1, x_2) dx_1 \right] dx_2.$$

7.25. (a) $I = \int_0^1 \left[\int_{1-x_2}^{\sqrt{1-x_2}} f(x_1, x_2) dx_1 \right] dx_2.$

(b) $dg/dx_1 = \int_{1-x_1}^{1-x_1^2} \frac{\partial f(x_1, x_2)}{\partial x_1} dx_2 - 2x_1 f(x_1, 1-x_1^2) + f(x_1, 1-x_1).$

7.26. Let $u = x_2^2/x_1$, $v = x_1^2/x_2$. Then $uv = x_1x_2$, and

$$\iint_D x_1x_2 dx_1 dx_2 = \int_1^2 \left[\int_1^2 uv \left| \frac{\partial(x_1, x_2)}{\partial(u, v)} \right| du \right] dv.$$

But

$$\left| \frac{\partial(x_1, x_2)}{\partial(u, v)} \right| = \left| \frac{\partial(u, v)}{\partial(x_1, x_2)} \right|^{-1} = \frac{1}{3}.$$

Hence,

$$\iint_D x_1x_2 dx_1 dx_2 = \frac{3}{4}.$$

7.28. Let $I(a) = \int_0^{\sqrt{3}} dx/(a+x^2)$. Then

$$\frac{d^2 I(a)}{da^2} = 2 \int_0^{\sqrt{3}} \frac{dx}{(a+x^2)^3}.$$

On the other hand,

$$\begin{aligned} I(a) &= \frac{1}{\sqrt{a}} \operatorname{Arctan} \sqrt{\frac{3}{a}} \\ \frac{d^2 I(a)}{da^2} &= \frac{3}{4} a^{-5/2} \operatorname{Arctan} \sqrt{\frac{3}{a}} + \frac{\sqrt{3}}{4} \frac{1}{a^2(a+3)} \\ &\quad + \frac{\sqrt{3}}{2} \frac{(2a+3)}{a^2(a+3)^2}. \end{aligned}$$

Thus

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{dx}{(a+x^2)^3} &= \frac{3}{8} a^{-5/2} \operatorname{Arctan} \sqrt{\frac{3}{a}} + \frac{\sqrt{3}}{8} \frac{1}{a^2(a+3)} \\ &\quad + \frac{\sqrt{3}}{4} \frac{(2a+3)}{a^2(a+3)^2}.\end{aligned}$$

Putting $a = 1$, we obtain

$$\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^3} = \frac{3}{8} \operatorname{Arctan} \sqrt{3} + \frac{7\sqrt{3}}{64}.$$

7.29. (a) The marginal density functions of X_1 and X_2 are

$$\begin{aligned}f_1(x_1) &= \int_0^1 (x_1 + x_2) dx_2 \\ &= x_1 + \frac{1}{2}, \quad 0 < x_1 < 1,\end{aligned}$$

$$\begin{aligned}f_2(x_2) &= \int_0^1 (x_1 + x_2) dx_1 \\ &= x_2 + \frac{1}{2}, \quad 0 < x_2 < 1,\end{aligned}$$

$$f(x_1, x_2) \neq f_1(x_1)f_2(x_2).$$

The random variables X_1 and X_2 are therefore not independent.

(b)

$$\begin{aligned}E(X_1 X_2) &= \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2) dx_1 dx_2 \\ &= \frac{1}{3}.\end{aligned}$$

7.30. The marginal densities of X_1 and X_2 are

$$\begin{aligned}f_1(x_1) &= \begin{cases} 1+x_1, & -1 < x_1 < 0, \\ 1-x_1, & 0 < x_1 < 1, \\ 0, & \text{otherwise,} \end{cases} \\ f_2(x_2) &= 2x_2, \quad 0 < x_2 < 1.\end{aligned}$$

Note that $f(x_1, x_2) \neq f_1(x_1)f_2(x_2)$. Hence, X_1 and X_2 are not independent. But

$$E(X_1 X_2) = \int_0^1 \left[\int_{-x_2}^{x_2} x_1 x_2 dx_1 \right] dx_2 = 0,$$

$$E(X_1) = \int_{-1}^0 x_1(1+x_1) dx_1 + \int_0^1 x_1(1-x_1) dx_1 = 0,$$

$$E(X_2) = \int_0^1 2x_2^2 dx_2 = \frac{2}{3}.$$

Hence,

$$E(X_1 X_2) = E(X_1)E(X_2) = 0.$$

7.31. (a) The joint density function of Y_1 and Y_2 is

$$g(y_1, y_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1} (1-y_1)^{\beta-1} y_2^{\alpha+\beta-1} e^{-y_2}.$$

(b) The marginal densities of Y_1 and Y_2 are, respectively,

$$g_1(y_1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1} (1-y_1)^{\beta-1}, \quad 0 < y_1 < 1,$$

$$g_2(y_2) = \frac{B(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha+\beta-1} e^{-y_2}, \quad 0 < y_2 < \infty.$$

(c) Since $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$,

$$g(y_1, y_2) = g_1(y_1)g_2(y_2),$$

and Y_1 and Y_2 are therefore independent.

7.32. Let $U = X_1$, $W = X_1 X_2$. Then, $X_1 = U$, $X_2 = W/U$. The joint density function of U and W is

$$g(u, w) = \frac{10w^2}{u^2}, \quad w < u < \sqrt{w}, \quad 0 < w < 1.$$

The marginal density function of W is

$$\begin{aligned} g_1(w) &= 10w^2 \int_w^{\sqrt{w}} \frac{du}{u^2} \\ &= 10w^2 \left(\frac{1}{w} - \frac{1}{\sqrt{w}} \right), \quad 0 < w < 1. \end{aligned}$$

7.34. The inverse of this transformation is

$$\begin{aligned} X_1 &= Y_1, \\ X_2 &= Y_2 - Y_1, \\ &\vdots \\ X_n &= Y_n - Y_{n-1}, \end{aligned}$$

and the absolute value of the Jacobian determinant of this transformation is 1. Therefore, the joint density function of the Y_i 's is

$$\begin{aligned} g(y_1, y_2, \dots, y_n) &= e^{-y_1} e^{-(y_2 - y_1)} \dots e^{-(y_n - y_{n-1})} \\ &= e^{-y_n}, \quad 0 < y_1 < y_2 < \dots < y_n < \infty. \end{aligned}$$

The marginal density function of Y_n is

$$\begin{aligned} g_n(y_n) &= e^{-y_n} \int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_2} dy_1 dy_2 \dots dy_{n-1} \\ &= \frac{y_n^{n-1}}{(n-1)!} e^{-y_n}, \quad 0 < y_n < \infty. \end{aligned}$$

7.37. Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)'$. The least-squares estimate of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y},$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)'$, and $\mathbf{X}$ is a matrix of order $n \times 3$ whose first column is the column of ones, and whose second and third columns are given by the values of x_i , x_i^2 , $i = 1, 2, \dots, n$.

7.38. Maximizing $L(\mathbf{x}, \mathbf{p})$ is equivalent to maximizing its natural logarithm. Let us therefore consider maximizing $\log L$ subject to $\sum_{i=1}^k p_i = 1$. Using the method of Lagrange multipliers, let

$$F = \log L + \lambda \left(\sum_{i=1}^k p_i - 1 \right).$$

Differentiating F with respect to $p_1, p_2, \dots, p_k$, and equating the derivatives to zero, we obtain, $x_i/p_i + \lambda = 0$ for $i = 1, 2, \dots, k$. Combining these equations with the constraint $\sum_{i=1}^k p_i = 1$, we obtain the maximum likelihood estimates

$$\hat{p}_i = \frac{x_i}{n}, \quad i = 1, 2, \dots, k.$$

CHAPTER 8

8.1. Let $\mathbf{x}_i = (x_{i1}, x_{i2})'$, $\mathbf{h}_i = (h_{i1}, h_{i2})'$, $i = 0, 1, 2, \dots$. Then

$$\begin{aligned} f(\mathbf{x}_i + t\mathbf{h}_i) &= 8(x_{i1} + th_{i1})^2 - 4(x_{i1} + th_{i1})(x_{i2} + th_{i2}) + 5(x_{i2} + th_{i2})^2 \\ &= a_i t^2 + b_i t + c_i, \end{aligned}$$

where

$$\begin{aligned} a_i &= 8h_{i1}^2 - 4h_{i1}h_{i2} + 5h_{i2}^2, \\ b_i &= 16x_{i1}h_{i1} - 4(x_{i1}h_{i2} + x_{i2}h_{i1}) + 10x_{i2}h_{i2}, \\ c_i &= 8x_{i1}^2 - 4x_{i1}x_{i2} + 5x_{i2}^2. \end{aligned}$$

The value of t that minimizes $f(\mathbf{x}_i + t\mathbf{h}_i)$ in the direction of $\mathbf{h}_i$ is given by the solution of $\partial f / \partial t = 0$, namely, $t_i = -b_i / (2a_i)$. Hence,

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{b_i}{2a_i} \mathbf{h}_i, \quad i = 0, 1, 2, \dots,$$

where

$$\mathbf{h}_i = - \frac{\nabla f(\mathbf{x}_i)}{\|\nabla f(\mathbf{x}_i)\|_2}, \quad i = 0, 1, 2, \dots,$$

and

$$\nabla f(\mathbf{x}_i) = (16x_{i1} - 4x_{i2}, -4x_{i1} + 10x_{i2})', \quad i = 0, 1, 2, \dots$$

The results of the iterative minimization procedure are given in the following table:

Iteration i	$\mathbf{x}_i$	$\mathbf{h}_i$	t_i	a_i	$f(\mathbf{x}_i)$
0	$(5, 2)'$	$(-1, 0)'$	4.5	8	180
1	$(0.5, 2)'$	$(0, -1)'$	1.8	5	18
2	$(0.5, 0.2)'$	$(-1, 0)'$	0.45	8	1.8
3	$(0.05, 0.2)'$	$(0, -1)'$	0.18	5	0.18
4	$(0.05, 0.02)'$	$(-1, 0)'$	0.045	8	0.018
5	$(0.005, 0.02)'$	$(0, -1)'$	0.018	5	0.0018
6	$(0.005, 0.002)'$	$(-1, 0)'$	0.0045	8	0.00018

Note that $a_i > 0$, which confirms that $t_i = -b_i/(2a_i)$ does indeed minimize $f(\mathbf{x}_i + t\mathbf{h}_i)$ in the direction of $\mathbf{h}_i$. It is obvious that if we were to continue with this iterative procedure, the point $\mathbf{x}_i$ would converge to $\mathbf{0}$.

8.3. (a)

$$\hat{y}(\mathbf{x}) = 45.690 + 4.919x_1 + 8.847x_2 \\ - 0.270x_1x_2 - 4.148x_1^2 - 4.298x_2^2.$$

(b) The matrix $\hat{\mathbf{B}}$ is

$$\hat{\mathbf{B}} = \begin{bmatrix} -4.148 & -0.135 \\ -0.135 & -4.298 \end{bmatrix}.$$

Its eigenvalues are $\tau_1 = -8.136$, $\tau_2 = -8.754$. This makes $\hat{\mathbf{B}}$ a negative definite matrix. The results of applying the method of ridge analysis inside the region R are given in the following table:

λ	x_1	x_2	r	$\hat{y}$
31.4126	0.0687	0.1236	0.1414	47.0340
13.5236	0.1373	0.2473	0.2829	48.2030
7.5549	0.2059	0.3709	0.4242	49.1969
4.5694	0.2745	0.4946	0.5657	50.0158
2.7797	0.3430	0.6184	0.7072	50.6595
1.5873	0.4114	0.7421	0.8485	51.1282
0.7359	0.4797	0.8659	0.9899	51.4218
0.0967	0.5480	0.9898	1.1314	51.5404
-0.4009	0.6163	1.1137	1.2729	51.4839
-0.7982	0.6844	1.2376	1.4142	51.2523

Note that the stationary point, $\mathbf{x}_0 = (0.5601, 1.0117)'$, is a point of absolute maximum since the eigenvalues of $\hat{\mathbf{B}}$ are negative. This point falls inside R . The corresponding maximum value of $\hat{y}$ is 51.5431.

8.4. We know that

$$\begin{aligned}(\hat{\mathbf{B}} - \lambda_1 \mathbf{I}_k) \mathbf{x}_1 &= -\frac{1}{2} \hat{\boldsymbol{\beta}}, \\ (\hat{\mathbf{B}} - \lambda_2 \mathbf{I}_k) \mathbf{x}_2 &= -\frac{1}{2} \hat{\boldsymbol{\beta}},\end{aligned}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ correspond to λ_1 and λ_2 , respectively, and are such that $\mathbf{x}'_1 \mathbf{x}_1 = \mathbf{x}'_2 \mathbf{x}_2 = r^2$, with r^2 being the common value of r_1^2 and r_2^2 . The corresponding values of $\hat{y}$ are

$$\begin{aligned}\hat{y}_1 &= \hat{\beta}_0 + \mathbf{x}'_1 \hat{\boldsymbol{\beta}} + \mathbf{x}'_1 \hat{\mathbf{B}} \mathbf{x}_1, \\ \hat{y}_2 &= \hat{\beta}_0 + \mathbf{x}'_2 \hat{\boldsymbol{\beta}} + \mathbf{x}'_2 \hat{\mathbf{B}} \mathbf{x}_2.\end{aligned}$$

We then have

$$\begin{aligned}\mathbf{x}'_1 \hat{\mathbf{B}} \mathbf{x}_1 - \mathbf{x}'_2 \hat{\mathbf{B}} \mathbf{x}_2 + \frac{1}{2}(\mathbf{x}'_1 - \mathbf{x}'_2) \hat{\boldsymbol{\beta}} &= (\lambda_1 - \lambda_2) r^2, \\ \hat{y}_1 - \hat{y}_2 &= \frac{1}{2}(\mathbf{x}'_1 - \mathbf{x}'_2) \hat{\boldsymbol{\beta}} + (\lambda_1 - \lambda_2) r^2.\end{aligned}$$

Furthermore, from the equations defining $\mathbf{x}_1$ and $\mathbf{x}_2$, we have

$$(\lambda_2 - \lambda_1) \mathbf{x}'_1 \mathbf{x}_2 = \frac{1}{2}(\mathbf{x}'_1 - \mathbf{x}'_2) \hat{\boldsymbol{\beta}}.$$

Hence,

$$\hat{y}_1 - \hat{y}_2 = (\lambda_1 - \lambda_2)(r^2 - \mathbf{x}'_1 \mathbf{x}_2).$$

But $r^2 = \|\mathbf{x}_1\|_2 \|\mathbf{x}_2\|_2 > \mathbf{x}'_1 \mathbf{x}_2$, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are not parallel vectors. We conclude that $\hat{y}_1 > \hat{y}_2$ whenever $\lambda_1 > \lambda_2$.

8.5. If $\mathbf{M}(\mathbf{x}_1)$ is positive definite, then λ_1 must be smaller than all the eigenvalues of $\hat{\mathbf{B}}$ (this is based on applying the spectral decomposition theorem to $\hat{\mathbf{B}}$ and using Theorem 2.3.12). Similarly, if $\mathbf{M}(\mathbf{x}_2)$ is indefinite, then λ_2 must be larger than the smallest eigenvalue of $\hat{\mathbf{B}}$ and also smaller than its largest eigenvalue. It follows that $\lambda_1 < \lambda_2$. Hence, by Exercise 8.4, $\hat{y}_1 < \hat{y}_2$.

8.6. (a) We know that

$$(\hat{\mathbf{B}} - \lambda \mathbf{I}_k) \mathbf{x} = -\frac{1}{2} \hat{\boldsymbol{\beta}}.$$

Differentiating with respect to λ gives

$$(\hat{\mathbf{B}} - \lambda \mathbf{I}_k) \frac{\partial \mathbf{x}}{\partial \lambda} = \mathbf{x},$$

and since $\mathbf{x}'\mathbf{x} = r^2$,

$$\mathbf{x}' \frac{\partial \mathbf{x}}{\partial \lambda} = r \frac{\partial r}{\partial \lambda}.$$

A second differentiation with respect to λ yields

$$\begin{aligned} (\hat{\mathbf{B}} - \lambda \mathbf{I}_k) \frac{\partial^2 \mathbf{x}}{\partial \lambda^2} &= 2 \frac{\partial \mathbf{x}}{\partial \lambda}, \\ \mathbf{x}' \frac{\partial^2 \mathbf{x}}{\partial \lambda^2} + \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} &= r \frac{\partial^2 r}{\partial \lambda^2} + \left(\frac{\partial r}{\partial \lambda} \right)^2. \end{aligned}$$

If we premultiply the second equation by $\partial^2 \mathbf{x}' / \partial \lambda^2$ and the fourth equation by $\partial \mathbf{x}' / \partial \lambda$, subtract, and transpose, we obtain

$$\mathbf{x}' \frac{\partial^2 \mathbf{x}}{\partial \lambda^2} - 2 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} = 0.$$

Substituting this in the fifth equation, we get

$$r \frac{\partial^2 r}{\partial \lambda^2} = 3 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} - \left(\frac{\partial r}{\partial \lambda} \right)^2.$$

Now, since

$$\begin{aligned} \frac{\partial r}{\partial \lambda} &= \frac{\partial}{\partial \lambda} (\mathbf{x}'\mathbf{x})^{1/2} \\ &= \mathbf{x}' \frac{\partial \mathbf{x} / \partial \lambda}{(\mathbf{x}'\mathbf{x})^{1/2}}, \end{aligned}$$

we conclude that

$$r^3 \frac{\partial^2 r}{\partial \lambda^2} = 3r^2 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} - \left(\mathbf{x}' \frac{\partial \mathbf{x}}{\partial \lambda} \right)^2.$$

(b) The expression in (a) can be written as

$$r^3 \frac{\partial^2 r}{\partial \lambda^2} = 2r^2 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} + \left[r^2 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda} - \left(\mathbf{x}' \frac{\partial \mathbf{x}}{\partial \lambda} \right)^2 \right].$$

The first part on the right-hand side is nonnegative and is zero only when $r = 0$ or when $\partial \mathbf{x} / \partial \lambda = \mathbf{0}$. The second part is nonnegative by

the fact that

$$\begin{aligned} \left(\mathbf{x}' \frac{\partial \mathbf{x}}{\partial \lambda} \right)^2 &\leq r^2 \left\| \frac{\partial \mathbf{x}}{\partial \lambda} \right\|_2^2 \\ &= r^2 \frac{\partial \mathbf{x}'}{\partial \lambda} \frac{\partial \mathbf{x}}{\partial \lambda}. \end{aligned}$$

Equality occurs only when $\mathbf{x} = \mathbf{0}$, that is, $r = 0$, or when $\partial \mathbf{x} / \partial \lambda = \mathbf{0}$. But when $\partial \mathbf{x} / \partial \lambda = \mathbf{0}$, we have $\mathbf{x} = \mathbf{0}$ if λ is different from all the eigenvalues of $\hat{\mathbf{B}}$, and thus $r = 0$. It follows that $\partial^2 r / \partial \lambda^2 > 0$ except when $r = 0$, where it takes the value zero.

8.7. (a)

$$\begin{aligned} B &= \frac{n\Omega}{\sigma^2} \int_R \{E[\hat{y}(\mathbf{x})] - \eta(\mathbf{x})\}^2 d\mathbf{x} \\ &= \frac{n}{\sigma^2} \{(\boldsymbol{\gamma} - \boldsymbol{\beta})' \boldsymbol{\Gamma}_{11} (\boldsymbol{\gamma} - \boldsymbol{\beta}) - 2(\boldsymbol{\gamma} - \boldsymbol{\beta})' \boldsymbol{\Gamma}_{12} \boldsymbol{\delta} + \boldsymbol{\delta}' \boldsymbol{\Gamma}_{22} \boldsymbol{\delta}\}, \end{aligned}$$

where $\boldsymbol{\Gamma}_{11}$, $\boldsymbol{\Gamma}_{12}$, $\boldsymbol{\Gamma}_{22}$ are the region moments defined in Section 8.4.3.

- (b) To minimize B we differentiate it with respect to $\boldsymbol{\gamma}$, equate the derivative to zero, and solve for $\boldsymbol{\gamma}$. We get

$$\begin{aligned} 2\boldsymbol{\Gamma}_{11}(\boldsymbol{\gamma} - \boldsymbol{\beta}) - 2\boldsymbol{\Gamma}_{12}\boldsymbol{\delta} &= \mathbf{0}, \\ \boldsymbol{\gamma} &= \boldsymbol{\beta} + \boldsymbol{\Gamma}_{11}^{-1} \boldsymbol{\Gamma}_{12} \boldsymbol{\delta} \\ &= \mathbf{C}\boldsymbol{\tau}. \end{aligned}$$

This solution minimizes B , since $\boldsymbol{\Gamma}_{11}$ is positive definite.

- (c) B is minimized if and only if $\boldsymbol{\gamma} = \mathbf{C}\boldsymbol{\tau}$. This is equivalent to stating that $\mathbf{C}\boldsymbol{\tau}$ is estimable, since $E(\hat{\boldsymbol{\lambda}}) = \boldsymbol{\gamma}$.
- (d) Writing $\hat{\boldsymbol{\lambda}}$ as a linear function of the vector $\mathbf{y}$ of observations of the form $\hat{\boldsymbol{\lambda}} = \mathbf{L}\mathbf{y}$, we obtain

$$\begin{aligned} E(\hat{\boldsymbol{\lambda}}) &= \mathbf{L}E(\mathbf{y}) \\ &= \mathbf{L}(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\delta}) \\ &= \mathbf{L}[\mathbf{X}:\mathbf{Z}]\boldsymbol{\tau}. \end{aligned}$$

But $\boldsymbol{\gamma} = E(\hat{\boldsymbol{\lambda}}) = \mathbf{C}\boldsymbol{\tau}$. We conclude that $\mathbf{C} = \mathbf{L}[\mathbf{X}:\mathbf{Z}]$.

- (e) It is obvious from part (d) that the rows of $\mathbf{C}$ are spanned by the rows of $[\mathbf{X}:\mathbf{Z}]$.

(f) The matrix $\mathbf{L}$ defined by $\mathbf{L} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ satisfies the equation

$$\mathbf{L}[\mathbf{X}:\mathbf{Z}] = \mathbf{C},$$

since

$$\begin{aligned}\mathbf{L}[\mathbf{X}:\mathbf{Z}] &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{X}:\mathbf{Z}] \\ &= [\mathbf{I}:(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}] \\ &= [\mathbf{I}:\mathbf{M}_{11}^{-1}\mathbf{M}_{12}] \\ &= [\mathbf{I}:\mathbf{\Gamma}_{11}^{-1}\mathbf{\Gamma}_{12}] \\ &= \mathbf{C}.\end{aligned}$$

8.8. If the region R is a sphere of radius 1, then $3g^2 \leq 1$. Now,

$$\mathbf{\Gamma}_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{5} \end{bmatrix},$$

$$\mathbf{\Gamma}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\mathbf{C} = [\mathbf{I}_4 : \mathbf{O}_{4 \times 3}]$. Furthermore,

$$\mathbf{X} = [\mathbf{1}_4 : \mathbf{D}],$$

$$\mathbf{Z} = \begin{bmatrix} g^2 & g^2 & g^2 \\ g^2 & -g^2 & -g^2 \\ -g^2 & g^2 & -g^2 \\ -g^2 & -g^2 & g^2 \end{bmatrix}.$$

Hence,

$$\begin{aligned}\mathbf{M}_{11} &= \frac{1}{4}\mathbf{X}'\mathbf{X} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & 0 \\ 0 & 0 & 0 & g^2 \end{bmatrix},\end{aligned}$$

$$\begin{aligned}\mathbf{M}_{12} &= \frac{1}{4}\mathbf{X}'\mathbf{Z} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -g^3 \\ 0 & -g^3 & 0 \\ -g^3 & 0 & 0 \end{bmatrix}.\end{aligned}$$

(a)

$$\mathbf{M}_{11} = \mathbf{\Gamma}_{11} \Rightarrow g^2 = \frac{1}{5}$$

$$\mathbf{M}_{12} = \mathbf{\Gamma}_{12} \Rightarrow g = 0$$

Thus, it is not possible to choose g so that $\mathbf{D}$ satisfies the conditions in (8.56).

(b) Suppose that there exists a matrix $\mathbf{L}$ of order 4×4 such that

$$\mathbf{C} = \mathbf{L}[\mathbf{X} : \mathbf{Z}].$$

Then

$$\mathbf{I}_4 = \mathbf{LX},$$

$$\mathbf{0} = \mathbf{LZ}.$$

The second equation implies that $\mathbf{L}$ is of rank 1, while the first equation implies that the rank of $\mathbf{L}$ is greater than or equal to the rank of $\mathbf{I}_4$, which is equal to 4. Therefore, it is not possible to find a matrix such as $\mathbf{L}$. Hence, g cannot be chosen so that $\mathbf{D}$ satisfies the minimum bias property described in part (e) of Exercise 8.7.

8.9. (a) Since $\mathbf{\Delta}$ is symmetric, it can be written as $\mathbf{\Delta} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$, where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues of $\mathbf{\Delta}$ and the columns of $\mathbf{P}$ are the corresponding orthogonal eigenvectors of $\mathbf{\Delta}$, each of length equal to 1 (see Theorem 2.3.10). It is easy to see that over the region ψ ,

$$h(\mathbf{\delta}, \mathbf{D}) \leq \mathbf{\delta}'\mathbf{\delta}e_{\max}(\mathbf{\Delta}) \leq r^2e_{\max}(\mathbf{\Delta}),$$

by the fact that $e_{\max}(\mathbf{\Delta})\mathbf{I} - \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$ is positive semidefinite. Without loss of generality, we consider that the diagonal elements of $\mathbf{\Lambda}$ are written in descending order. The upper bound in the above inequality is attained by $h(\mathbf{\delta}, \mathbf{D})$ for $\mathbf{\delta} = r\mathbf{P}_1$, where $\mathbf{P}_1$ is the first column of $\mathbf{P}$, which corresponds to $e_{\max}(\mathbf{\Delta})$.

(b) The design $\mathbf{D}$ can be chosen so that $e_{\max}(\mathbf{\Delta})$ is minimized over the region R .

8.10. This is similar to Exercise 8.9. Write $\mathbf{T}$ as $\mathbf{P}\mathbf{\Lambda}\mathbf{P}'$, where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues of $\mathbf{T}$, and $\mathbf{P}$ is an orthogonal matrix of corresponding eigenvectors. Then $\mathbf{\delta}'\mathbf{T}\mathbf{\delta} = \mathbf{u}'\mathbf{u}$, where $\mathbf{u} = \mathbf{\Lambda}^{1/2}\mathbf{P}'\mathbf{\delta}$, and

$\lambda(\delta, \mathbf{D}) = \delta' \mathbf{S} \delta = \mathbf{u}' \mathbf{\Lambda}^{-1/2} \mathbf{P}' \mathbf{S} \mathbf{P} \mathbf{\Lambda}^{-1/2} \mathbf{u}$. Hence, over the region Φ ,

$$\delta' \mathbf{S} \delta \geq \kappa e_{\min}(\mathbf{\Lambda}^{-1/2} \mathbf{P}' \mathbf{S} \mathbf{P} \mathbf{\Lambda}^{-1/2}).$$

But, if $\mathbf{S}$ is positive definite, then by Theorem 2.3.9,

$$\begin{aligned} e_{\min}(\mathbf{\Lambda}^{-1/2} \mathbf{P}' \mathbf{S} \mathbf{P} \mathbf{\Lambda}^{-1/2}) &= e_{\min}(\mathbf{P} \mathbf{\Lambda}^{-1} \mathbf{P}' \mathbf{S}) \\ &= e_{\min}(\mathbf{T}^{-1} \mathbf{S}). \end{aligned}$$

8.11. (a) Using formula (8.52), it can be shown that (see Khuri and Cornell, 1996, page 229)

$$\begin{aligned} V = \frac{1}{\lambda_4(k+2) - \lambda_2^2 k} &\left[\lambda_4(k+2) - 2k \left(\lambda_2 - \frac{\lambda_4}{\lambda_2} \right) \right. \\ &\left. + 2k \frac{\lambda_4(k+1) - \lambda_2^2(k-1)}{\lambda_4(k+4)} \right], \end{aligned}$$

where k is the number of input variables (that is, $k = 2$),

$$\begin{aligned} \lambda_2 &= \frac{1}{n} \sum_{u=1}^n x_{ui}^2, \quad i = 1, 2 \\ &= \frac{1}{n} (2^k + 2\alpha^2) \\ &= \frac{8}{n}, \\ \lambda_4 &= \frac{1}{n} \sum_{u=1}^n x_{ui}^2 x_{uj}^2, \quad i \neq j \\ &= \frac{2^k}{n} \\ &= \frac{4}{n}, \end{aligned}$$

and $n = 2^k + 2k + n_0 = 8 + n_0$ is the total number of observations. Here, x_{ui} denotes the design setting of variable x_i , $i = 1, 2$; $u = 1, 2, \dots, n$.

(b) The quantity V , being a function of n_0 , can be minimized with respect to n_0 .

8.12. We have that

$$\begin{aligned} (\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB}) &= (\mathbf{Y} - \mathbf{XB} + \mathbf{XB} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB} + \mathbf{XB} - \mathbf{XB}) \\ &= (\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB}) + (\mathbf{XB} - \mathbf{XB})'(\mathbf{XB} - \mathbf{XB}), \end{aligned}$$

since

$$(\mathbf{Y} - \mathbf{XB})'(\mathbf{XB} - \mathbf{XB}) = (\mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{XB})'(\mathbf{B} - \mathbf{B}) = \mathbf{0}.$$

Furthermore, since $(\mathbf{XB} - \mathbf{XB})'(\mathbf{XB} - \mathbf{XB})$ is positive semidefinite, then by Theorem 2.3.19,

$$e_i[(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})] \geq e_i[(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})], \quad i = 1, 2, \dots, r,$$

where $e_i(\cdot)$ denotes the i th eigenvalue of a square matrix. If $(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})$ and $(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})$ are nonsingular, then by multiplying the eigenvalues on both sides of the inequality, we obtain

$$\det[(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})] \geq \det[(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})].$$

Equality holds when $\mathbf{B} = \hat{\mathbf{B}}$.

8.13. For any b , $1 + b \leq e^b$. Let $a = 1 + b$. Then $a \leq e^{a-1}$. Now, let λ_i denote the i th eigenvalue of $\mathbf{A}$. Then $\lambda_i \geq 0$, and by the previous inequality,

$$\prod_{i=1}^p \lambda_i \leq \exp\left(\sum_{i=1}^p \lambda_i - p\right).$$

Hence,

$$\det(\mathbf{A}) \leq \exp[\text{tr}(\mathbf{A} - \mathbf{I}_p)].$$

8.14. The likelihood function is proportional to

$$[\det(\mathbf{V})]^{-n/2} \exp\left[-\frac{n}{2} \text{tr}(\mathbf{SV}^{-1})\right].$$

Now, by Exercise 8.13,

$$\det(\mathbf{SV}^{-1}) \leq \exp[\text{tr}(\mathbf{SV}^{-1} - \mathbf{I}_p)],$$

since $\det(\mathbf{S}\mathbf{V}^{-1}) = \det(\mathbf{V}^{-1/2}\mathbf{S}\mathbf{V}^{-1/2})$, $\text{tr}(\mathbf{S}\mathbf{V}^{-1}) = \text{tr}(\mathbf{V}^{-1/2}\mathbf{S}\mathbf{V}^{-1/2})$, and $\mathbf{V}^{-1/2}\mathbf{S}\mathbf{V}^{-1/2}$ is positive semidefinite. Hence,

$$[\det(\mathbf{S}\mathbf{V}^{-1})]^{n/2} \exp\left[-\frac{n}{2}\text{tr}(\mathbf{S}\mathbf{V}^{-1})\right] \leq \exp\left[-\frac{n}{2}\text{tr}(\mathbf{I}_p)\right].$$

This results in the following inequality:

$$[\det(\mathbf{V})]^{-n/2} \exp\left[-\frac{n}{2}\text{tr}(\mathbf{S}\mathbf{V}^{-1})\right] \leq [\det(\mathbf{S})]^{-n/2} \exp\left[-\frac{n}{2}\text{tr}(\mathbf{I}_p)\right],$$

which is the desired result.

- 8.16.** $\hat{y}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, and $\mathbf{f}'(\mathbf{x})$ is as in model (8.47). Simultaneous $(1 - \alpha) \times 100\%$ confidence intervals on $\mathbf{f}'(\mathbf{x})\hat{\boldsymbol{\beta}}$ for all $\mathbf{x}$ in R are of the form

$$\mathbf{f}'(\mathbf{x})\hat{\boldsymbol{\beta}} \mp (pMS_E F_{\alpha, p, n-p})^{1/2} [\mathbf{f}'(\mathbf{x})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{f}(\mathbf{x})]^{1/2}.$$

For the points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$, the joint confidence coefficient is at least $1 - \alpha$.

CHAPTER 9

- 9.1.** If $f(x)$ has a continuous derivative on $[0, 1]$, then by Theorems 3.4.5 and 4.2.2 we can find a positive constant A such that

$$|f(x_1) - f(x_2)| \leq A|x_1 - x_2|$$

for all x_1, x_2 in $[0, 1]$. Thus, by Definition 9.1.2,

$$\omega(\delta) \leq A\delta.$$

Using now Theorem 9.1.3, we obtain

$$|f(x) - b_n(x)| \leq \frac{3}{2} \frac{A}{n^{1/2}}$$

for all x in $[0, 1]$. Hence,

$$\sup_{0 \leq x \leq 1} |f(x) - b_n(x)| \leq \frac{c}{n^{1/2}},$$

where $c = \frac{3}{2}A$.

9.2. We have that

$$|f(x_1) - f(x_2)| \leq \sup_{|z_1 - z_2| \leq \delta_2} |f(z_1) - f(z_2)|$$

for all $|x_1 - x_2| \leq \delta_2$, and hence for all $|x_1 - x_2| \leq \delta_1$, since $\delta_1 \leq \delta_2$. It follows that

$$\sup_{|x_1 - x_2| \leq \delta_1} |f(x_1) - f(x_2)| \leq \sup_{|z_1 - z_2| \leq \delta_2} |f(z_1) - f(z_2)|,$$

that is, $\omega(\delta_1) \leq \omega(\delta_2)$.

9.3. Suppose that $f(x)$ is uniformly continuous on $[a, b]$. Then, for a given $\epsilon > 0$, there exists a positive $\delta(\epsilon)$ such that $|f(x_1) - f(x_2)| < \epsilon$ for all x_1, x_2 in $[a, b]$ for which $|x_1 - x_2| < \delta$. This implies that $\omega(\delta) \leq \epsilon$ and hence $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Vice versa, if $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, then for a given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $\omega(\delta) < \epsilon$ if $\delta < \delta_1$. This implies that

$$|f(x_1) - f(x_2)| < \epsilon \quad \text{if } |x_1 - x_2| \leq \delta < \delta_1,$$

and $f(x)$ must therefore be uniformly continuous on $[a, b]$.

9.4. By Theorem 9.1.1, there exists a sequence of polynomials, namely the Bernstein polynomials $\{b_n(x)\}_{n=1}^{\infty}$, that converges to $f(x) = |x|$ uniformly on $[-a, a]$. Let $p_n(x) = b_n(x) - b_n(0)$. Then $p_n(0) = 0$, and $p_n(x)$ converges uniformly to $|x|$ on $[-a, a]$, since $b_n(0) \rightarrow 0$ as $n \rightarrow \infty$.

9.5. The stated condition implies that $\int_0^1 f(x)p_n(x) dx = 0$ for any polynomial $p_n(x)$ of degree n . In particular, if we choose $p_n(x)$ to be a Bernstein polynomial for $f(x)$, then it will converge uniformly to $f(x)$ on $[0, 1]$. By Theorem 6.6.1,

$$\begin{aligned} \int_0^1 [f(x)]^2 dx &= \lim_{n \rightarrow \infty} \int_0^1 f(x)p_n(x) dx \\ &= 0. \end{aligned}$$

Since $f(x)$ is continuous, it must be zero everywhere on $[0, 1]$. If not, then by Theorem 3.4.3, there exists a neighborhood of a point in $[0, 1]$ [at which $f(x) \neq 0$] on which $f(x) \neq 0$. This causes $\int_0^1 [f(x)]^2 dx$ to be positive, a contradiction.

9.6. Using formula (9.15), we have

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \prod_{i=0}^n (x - a_i).$$

But $|\prod_{i=0}^n (x - a_i)| \leq n!(h^{n+1}/4)$ (see Prenter, 1975, page 37). Hence,

$$\sup_{a \leq x \leq b} |f(x) - p(x)| \leq \frac{\tau_{n+1} h^{n+1}}{4(n+1)}.$$

9.7. Using formula (9.14) with a_0, a_1, a_2 , and a_3 , we obtain

$$p(x) = \ell_0(x) \log a_0 + \ell_1(x) \log a_1 + \ell_2(x) \log a_2 + \ell_3(x) \log a_3,$$

where

$$\ell_0(x) = \left(\frac{x - a_1}{a_0 - a_1} \right) \left(\frac{x - a_2}{a_0 - a_2} \right) \left(\frac{x - a_3}{a_0 - a_3} \right),$$

$$\ell_1(x) = \left(\frac{x - a_0}{a_1 - a_0} \right) \left(\frac{x - a_2}{a_1 - a_2} \right) \left(\frac{x - a_3}{a_1 - a_3} \right),$$

$$\ell_2(x) = \left(\frac{x - a_0}{a_2 - a_0} \right) \left(\frac{x - a_1}{a_2 - a_1} \right) \left(\frac{x - a_3}{a_2 - a_3} \right),$$

$$\ell_3(x) = \left(\frac{x - a_0}{a_3 - a_0} \right) \left(\frac{x - a_1}{a_3 - a_1} \right) \left(\frac{x - a_2}{a_3 - a_2} \right).$$

Values of $f(x) = \log x$ and the corresponding values of $p(x)$ at several points inside the interval $[3.50, 3.80]$ are given below:

x	$f(x)$	$p(x)$
3.50	1.25276297	1.25276297
3.52	1.25846099	1.25846087
3.56	1.26976054	1.26976043
3.60	1.28093385	1.28093385
3.62	1.28647403	1.28647407
3.66	1.29746315	1.29746322
3.70	1.30833282	1.30833282
3.72	1.31372367	1.31372361
3.77	1.327075	1.32707487
3.80	1.33500107	1.33500107

Using the result of Exercise 9.6, an upper bound on the error of approximation is given by

$$\sup_{3.5 \leq x \leq 3.8} |f(x) - p(x)| \leq \frac{\tau_4 h^4}{16},$$

where $h = \max_i(a_{i+1} - a_i) = 0.10$, and

$$\begin{aligned} \tau_4 &= \sup_{3.5 \leq x \leq 3.8} |f^{(4)}(x)| \\ &= \sup_{3.5 \leq x \leq 3.8} \left| \frac{-6}{x^4} \right| \\ &= \frac{6}{(3.5)^4}. \end{aligned}$$

Hence, the desired upper bound is

$$\begin{aligned} \frac{\tau_4 h^4}{16} &= \frac{6}{16} \left(\frac{0.10}{3.5} \right)^4 \\ &= 2.5 \times 10^{-7}. \end{aligned}$$

9.8. We have that

$$\begin{aligned} \int_a^b [f''(x) - s''(x)]^2 dx &= \int_a^b [f''(x)]^2 dx - \int_a^b [s''(x)]^2 dx \\ &\quad - 2 \int_a^b s''(x) [f''(x) - s''(x)] dx. \end{aligned}$$

But integration by parts yields

$$\begin{aligned} &\int_a^b s''(x) [f''(x) - s''(x)] dx \\ &= s''(x) [f'(x) - s'(x)] \Big|_a^b - \int_a^b s'''(x) [f'(x) - s'(x)] dx. \end{aligned}$$

The first term on the right-hand side is zero, since $f'(x) = s'(x)$ at $x = a, b$; and the second term is also zero, by the fact that $s'''(x)$ is a constant, say $s'''(x) = \lambda_i$, over (τ_i, τ_{i+1}) . Hence,

$$\int_a^b s'''(x) [f'(x) - s'(x)] dx = \sum_{i=0}^{n-1} \lambda_i \int_{\tau_i}^{\tau_{i+1}} [f'(x) - s'(x)] dx = 0.$$

It follows that

$$\int_a^b [f''(x) - s''(x)]^2 dx = \int_a^b [f''(x)]^2 dx - \int_a^b [s''(x)]^2 dx,$$

which implies the desired result.

9.10.

$$\frac{\partial^{r+1} h(x, \boldsymbol{\theta})}{\partial x^{r+1}} = (-1)^r \theta_1 \theta_2 \theta_3^{r+1} e^{-\theta_3 x}.$$

Hence,

$$\sup_{0 \leq x \leq 8} \left| \frac{\partial^{r+1} h(x, \boldsymbol{\theta})}{\partial x^{r+1}} \right| \leq 50.$$

Using inequality (9.36), the integer r is determined such that

$$\frac{2}{(r+1)!} \left(\frac{8}{4} \right)^{r+1} (50) < 0.05.$$

The smallest integer that satisfies this inequality is $r = 10$. The Chebyshev points corresponding to this value of r are $z_0, z_1, \dots, z_{10}$, where by formula (9.18),

$$z_i = 4 + 4 \cos \left[\left(\frac{2i+1}{22} \right) \pi \right], \quad i = 0, 1, \dots, 10.$$

Using formula (9.37), the Lagrange interpolating polynomial that approximates $h(x, \boldsymbol{\theta})$ over $[0, 8]$ is given by

$$p_{10}(x, \boldsymbol{\theta}) = \theta_1 \sum_{i=0}^{10} [1 - \theta_2 e^{-\theta_3 z_i}] \mathcal{L}_i(x),$$

where $\mathcal{L}_i(x)$ is a polynomial of degree 10 which can be obtained from (9.13) by substituting z_i for a_i ($i = 0, 1, \dots, 10$).

9.11. We have that

$$\frac{\partial^4 \eta(x, \alpha, \beta)}{\partial x^4} = (0.49 - \alpha) \beta^4 e^{8\beta} e^{-\beta x}.$$

Hence,

$$\max_{10 \leq x \leq 40} \left| \frac{\partial^4 \eta(x, \alpha, \beta)}{\partial x^4} \right| = (0.49 - \alpha) \beta^4 e^{-2\beta}.$$

It can be verified that the function $f(\beta) = \beta^4 e^{-2\beta}$ is strictly monotone increasing for $0.06 \leq \beta \leq 0.16$. Therefore, $\beta = 0.16$ maximizes $f(\beta)$

over this interval. Hence,

$$\begin{aligned} \max_{10 \leq x \leq 40} \left| \frac{\partial^4 \eta(x, \alpha, \beta)}{\partial x^4} \right| &\leq (0.49 - \alpha)(0.16)^4 e^{-0.32} \\ &\leq 0.13(0.16)^4 e^{-0.32} \\ &= 0.0000619, \end{aligned}$$

since $0.36 \leq \alpha \leq 0.41$. Using Theorem 9.3.1, we have

$$\max_{10 \leq x \leq 40} |\eta(x, \alpha, \beta) - s(x)| \leq \frac{5}{384} \Delta^4 (0.0000619).$$

Here, we considered equally spaced partition points with $\Delta\tau_i = \Delta$. Let us now choose Δ such that

$$\frac{5}{384} \Delta^4 (0.0000619) < 0.001.$$

This is satisfied if $\Delta < 5.93$. Choosing $\Delta = 5$, the number of knots needed is

$$\begin{aligned} m &= \frac{40 - 10}{\Delta} - 1 \\ &= 5. \end{aligned}$$

9.12.

$$\det(\mathbf{X}'\mathbf{X}) = [(x_3 - \alpha)^2(x_2 - x_1) - (x_3 - x_1)(x_2 - \alpha)^2]^2.$$

The determinant is maximized when $x_1 = -1$, $x_2 = \frac{1}{4}(1 + \alpha)^2$, $x_3 = 1$.

CHAPTER 10

10.1. We have that

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x] dx \\ &= \begin{cases} 0, & n \neq m, \\ 1, & n = m \geq 1, \end{cases} \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \sin mx dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sin(n+m)x - \sin(n-m)x] dx \\ &= 0 \quad \text{for all } m, n, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] dx \\ &= \begin{cases} 0, & n \neq m, \\ 1, & n = m \geq 1. \end{cases} \end{aligned}$$

10.2. (a)

- (i) Integrating n times by parts [x^m is differentiated and $p_n(x)$ is integrated], we obtain

$$\int_{-1}^1 x^m p_n(x) dx = 0 \quad \text{for } m = 0, 1, \dots, n-1.$$

- (ii) Integrating n times by parts results in

$$\int_{-1}^1 x^n p_n(x) dx = \frac{1}{2^n} \int_{-1}^1 (1-x^2)^n dx.$$

Letting $x = \cos \theta$, we obtain

$$\begin{aligned} \int_{-1}^1 (1-x^2)^n dx &= \int_0^\pi \sin^{2n+1} \theta d\theta \\ &= \frac{2^{2n+1}}{2n+1} \left(\frac{2n}{n} \right)^{-1}, \quad n \geq 0. \end{aligned}$$

- (b) This is obvious, since (a) is true for $m = 0, 1, \dots, n-1$.

(c)

$$\int_{-1}^1 p_n^2(x) dx = \int_{-1}^1 \left[\frac{1}{2^n} \left(\frac{2n}{n} \right) x^n + \pi_{n-1}(x) \right] p_n(x) dx,$$

where $\pi_{n-1}(x)$ denotes a polynomial of degree $n-1$. Hence, using the results in (a) and (b), we obtain

$$\begin{aligned} \int_{-1}^1 p_n^2(x) dx &= \frac{1}{2^n} \left(\frac{2n}{n} \right) \frac{2^{n+1}}{2n+1} \left(\frac{2n}{n} \right)^{-1} \\ &= \frac{2}{2n+1}, \quad n \geq 0. \end{aligned}$$

10.3. (a)

$$\begin{aligned} T_n(\zeta_i) &= \cos \left\{ n \operatorname{Arccos} \left[\cos \frac{(2i-1)\pi}{2n} \right] \right\} \\ &= \cos \frac{(2i-1)\pi}{2} = 0 \end{aligned}$$

for $i = 1, 2, \dots, n$.

- (b) $T'_n(x) = (n/\sqrt{1-x^2}) \sin(n \operatorname{Arccos} x)$. Hence,

$$T'_n(\zeta_i) = \frac{n}{\sqrt{1-\zeta_i^2}} \sin \frac{(2i-1)\pi}{2} \neq 0$$

for $i = 1, 2, \dots, n$.

10.4. (a)

$$\frac{dH_n(x)}{dx} = (-1)^n x e^{x^2/2} \frac{d^n(e^{-x^2/2})}{dx^n} + (-1)^n e^{x^2/2} \frac{d^{n+1}(e^{-x^2/2})}{dx^{n+1}}.$$

Using formulas (10.21) and (10.24), we have

$$\begin{aligned} \frac{dH_n(x)}{dx} &= (-1)^n x e^{x^2/2} (-1)^n e^{-x^2/2} H_n(x) \\ &\quad + (-1)^n e^{x^2/2} (-1)^{n+1} e^{-x^2/2} H_{n+1}(x) \\ &= xH_n(x) - H_{n+1}(x) \\ &= nH_{n-1}(x), \quad \text{by formula (10.25).} \end{aligned}$$

(b) From (10.23) and (10.24), we have

$$H_{n+1}(x) = xH_n(x) - \frac{dH_n(x)}{dx}.$$

Hence,

$$\begin{aligned} \frac{dH_{n+1}(x)}{dx} &= x \frac{dH_n(x)}{dx} + H_n(x) - \frac{d^2H_n(x)}{dx^2} \\ \frac{d^2H_n(x)}{dx^2} &= x \frac{dH_n(x)}{dx} + H_n(x) - \frac{dH_{n+1}(x)}{dx} \\ &= x \frac{dH_n(x)}{dx} + H_n(x) - (n+1)H_n(x), \quad \text{by (a)} \\ &= x \frac{dH_n(x)}{dx} - nH_n(x), \end{aligned}$$

which gives the desired result.

10.5. (a) Using formula (10.18), we can show that

$$\left| \frac{\sin[(n+1)\theta]}{\sin \theta} \right| \leq n+1$$

by mathematical induction. Obviously, the inequality is true for $n = 1$, since

$$\left| \frac{\sin 2\theta}{\sin \theta} \right| = \left| \frac{2 \sin \theta \cos \theta}{\sin \theta} \right| \leq 2.$$

Suppose now that the inequality is true for $n = m$. To show that it is true for $n = m + 1$ ($m \geq 1$):

$$\left| \frac{\sin[(m+2)\theta]}{\sin \theta} \right| = \left| \frac{\sin[(m+1)\theta]\cos \theta + \cos[(m+1)\theta]\sin \theta}{\sin \theta} \right| \leq m + 1 + 1 = m + 2.$$

Therefore, the inequality is true for all n .

(b) From Section 10.4.2 we have that

$$\frac{dT_n(x)}{dx} = n \frac{\sin n\theta}{\sin \theta}$$

Hence,

$$\left| \frac{dT_n(x)}{dx} \right| = n \left| \frac{\sin n\theta}{\sin \theta} \right| \leq n^2,$$

since $|\sin n\theta/\sin \theta| \leq n$, which can be proved by induction as in (a). Note that as $x \rightarrow \mp 1$, that is, as $\theta \rightarrow 0$ or $\theta \rightarrow \pi$,

$$\left| \frac{dT_n(x)}{dx} \right| \rightarrow n^2.$$

(c) Making the change of variable $t = \cos \theta$, we get

$$\begin{aligned} \int_{-1}^x \frac{T_n(t)}{\sqrt{1-t^2}} dt &= - \int_{\pi}^{\text{Arccos } x} \cos n\theta d\theta \\ &= - \frac{1}{n} \sin n\theta \Big|_{\pi}^{\text{Arccos } x} \\ &= - \frac{1}{n} \sin(n \text{Arccos } x) \\ &= - \frac{1}{n} \sin n\psi, \quad \text{where } x = \cos \psi \\ &= - \frac{1}{n} \sin \psi U_{n-1}(x), \quad \text{by (10.18)} \\ &= - \frac{1}{n} \sqrt{1-x^2} U_{n-1}(x). \end{aligned}$$

10.7. The first two Laguerre polynomials are $L_0(x) = 1$ and $L_1(x) = x - \alpha - 1$, as can be seen from applying the Rodrigues formula in Section 10.6. Now, differentiating $H(x, t)$ with respect to t , we obtain

$$\frac{\partial H(x, t)}{\partial t} = \left[\frac{\alpha + 1}{1 - t} - \frac{x}{(1 - t)^2} \right] H(x, t),$$

or equivalently,

$$(1 - t)^2 \frac{\partial H(x, t)}{\partial t} + [(x - \alpha - 1) + (\alpha + 1)t] H(x, t) = 0.$$

Hence, $g_0(x) = H(x, 0) = 1$, and $g_1(x) = -\partial H(x, t)/\partial t|_{t=0} = x - \alpha - 1$. Thus, $g_0(x) = L_0(x)$ and $g_1(x) = L_1(x)$. Furthermore, if the representation

$$H(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} g_n(x) t^n$$

is substituted in the above equation, and if the coefficient of t^n in the resulting series is equated to zero, we obtain

$$g_{n+1}(x) + (2n - x + \alpha + 1)g_n(x) + (n^2 + n\alpha)g_{n-1}(x) = 0.$$

This is the same relation connecting the Laguerre polynomials $L_{n+1}(x)$, $L_n(x)$, and $L_{n-1}(x)$, which is given at the end of Section 10.6. Since we have already established that $g_n(x) = L_n(x)$ for $n = 0, 1$, we conclude that the same relation holds for all values of n .

10.8. Using formula (10.40), we have

$$\begin{aligned} p_4^*(x) &= c_0 + c_1 x + c_2 \left(\frac{3x^2}{2} - \frac{1}{2} \right) \\ &\quad + c_3 \left(\frac{5x^3}{2} - \frac{3x}{2} \right) + c_4 \left(\frac{35x^4}{8} - \frac{30x^2}{8} + \frac{3}{8} \right), \end{aligned}$$

where

$$\begin{aligned} c_0 &= \frac{1}{2} \int_{-1}^1 e^x dx, \\ c_1 &= \frac{3}{2} \int_{-1}^1 x e^x dx, \\ c_2 &= \frac{5}{2} \int_{-1}^1 \left(\frac{3x^2}{2} - \frac{1}{2} \right) e^x dx, \end{aligned}$$

$$c_3 = \frac{7}{2} \int_{-1}^1 \left(\frac{5x^3}{2} - \frac{3x}{2} \right) e^x dx,$$

$$c_4 = \frac{9}{2} \int_{-1}^1 \left(\frac{35x^4}{8} - \frac{30x^2}{8} + \frac{3}{8} \right) e^x dx.$$

In computing c_0, c_1, c_2, c_3, c_4 we have made use of the fact that

$$\int_{-1}^1 p_n^2(x) dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \dots,$$

where $p_n(x)$ is the Legendre polynomial of degree n (see Section 10.2.1).

10.9.

$$\begin{aligned} f(x) &\approx \frac{c_0}{2} + c_1 T_1(x) + c_2 T_2(x) + c_3 T_3(x) + c_4 T_4(x) \\ &= 1.266066 + 1.130318 T_1(x) + 0.271495 T_2(x) \\ &\quad + 0.044337 T_3(x) + 0.005474 T_4(x) \\ &= 1.000044 + 0.99731x + 0.4992x^2 + 0.177344x^3 + 0.043792x^4 \end{aligned}$$

[Note: For more computational details, see Example 7.2 in Ralston and Rabinowitz (1978).]

10.10. Use formula (10.48) with the given values of the central moments.

10.11. The first six cumulants of the standardized chi-squared distribution with five degrees of freedom are $\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = 1.264911064$, $\kappa_4 = 2.40$, $\kappa_5 = 6.0715731$, $\kappa_6 = 19.2$. We also have that $z_{0.05} = 1.645$. Applying the Cornish-Fisher approximation for $x_{0.05}$, we obtain the value $x_{0.05} \approx 1.921$. Thus, $P(\chi_5^{*2} > 1.921) \approx 0.05$, where χ_5^{*2} denotes the standardized chi-squared variate with five degrees of freedom. If χ_5^2 denotes the nonstandardized chi-squared counterpart (with five degrees of freedom), then $\chi_5^2 = \sqrt{10} \chi_5^{*2} + 5$. Hence, the corresponding approximate value of the upper 0.05 quantile of χ_5^2 is $\sqrt{10}(1.921) + 5 = 11.0747$. The actual table value is 11.07.

10.12. (a)

$$\begin{aligned} \int_0^1 e^{-t^2/2} dt &\approx 1 - \frac{1}{2 \times 3 \times 1!} + \frac{1}{2^2 \times 5 \times 2!} - \frac{1}{2^3 \times 7 \times 3!} \\ &\quad + \frac{1}{2^4 \times 9 \times 4!} = 0.85564649. \end{aligned}$$

(b)

$$\int_0^x e^{-t^2/2} dt \approx x e^{-x^2/8} \left[\Theta_0\left(\frac{x}{2}\right) + \frac{1}{3} \Theta_2\left(\frac{x}{2}\right) + \frac{1}{5} \Theta_4\left(\frac{x}{2}\right) + \frac{1}{7} \Theta_6\left(\frac{x}{2}\right) \right],$$

where

$$\begin{aligned} \Theta_0\left(\frac{x}{2}\right) &= \frac{1}{1!} H_0\left(\frac{x}{2}\right) \\ &= 1, \\ \Theta_2\left(\frac{x}{2}\right) &= \frac{1}{2!} \left(\frac{x}{2}\right)^2 H_2\left(\frac{x}{2}\right) \\ &= \frac{1}{2!} \left(\frac{x}{2}\right)^2 \left[\left(\frac{x}{2}\right)^2 - 1 \right], \\ \Theta_4\left(\frac{x}{2}\right) &= \frac{1}{4!} \left(\frac{x}{2}\right)^4 \left[\left(\frac{x}{2}\right)^4 - 6\left(\frac{x}{2}\right)^2 + 3 \right], \\ \Theta_6\left(\frac{x}{2}\right) &= \frac{1}{6!} \left(\frac{x}{2}\right)^6 \left[\left(\frac{x}{2}\right)^6 - 15\left(\frac{x}{2}\right)^4 + 45\left(\frac{x}{2}\right)^2 - 15 \right]. \end{aligned}$$

Hence,

$$\int_0^1 e^{-t^2/2} dt \approx 0.85562427.$$

10.13. Using formula (10.21), we have that

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} b_n (-1)^n e^{x^2/2} \frac{d^n(e^{-x^2/2})}{dx^n} \phi(x) \\ &= \sum_{n=0}^{\infty} (-1)^n b_n \frac{1}{\sqrt{2\pi}} \frac{d^n(e^{-x^2/2})}{dx^n}, \quad \text{by (10.44)} \\ &= \sum_{n=0}^{\infty} \frac{c_n}{n!} \frac{d^n \phi(x)}{dx^n}, \end{aligned}$$

where $c_n = (-1)^n n! b_n$, $n = 0, 1, \dots$

10.14. The moment generating function of any one of the X_i^2 's is

$$\begin{aligned}\psi(t) &= \int_{-\infty}^{\infty} e^{tx^2} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{tx^2} \left[\phi(x) - \frac{\lambda_3}{6} \frac{d^3\phi(x)}{dx^3} + \frac{\lambda_4}{24} \frac{d^4\phi(x)}{dx^4} + \frac{\lambda_3^2}{72} \frac{d^6\phi(x)}{dx^6} \right] dx.\end{aligned}$$

By formula (10.21),

$$\frac{d^n\phi(x)}{dx^n} = (-1)^n \phi(x) H_n(x), \quad n = 0, 1, 2, \dots$$

Hence,

$$\begin{aligned}\psi(t) &= \int_{-\infty}^{\infty} e^{tx^2} \left[\phi(x) + \frac{\lambda_3}{6} \phi(x) H_3(x) + \frac{\lambda_4}{24} \phi(x) H_4(x) \right. \\ &\quad \left. + \frac{\lambda_3^2}{72} \phi(x) H_6(x) \right] dx \\ &= 2 \int_0^{\infty} e^{tx^2} \left[\phi(x) + \frac{\lambda_4}{24} \phi(x) H_4(x) + \frac{\lambda_3^2}{72} \phi(x) H_6(x) \right] dx,\end{aligned}$$

since $H_3(x)$ is an odd function, and $H_4(x)$ and $H_6(x)$ are even functions. It is known that

$$\int_0^{\infty} e^{-x^2/2} x^{2n-1} dx = 2^{n-1} \Gamma(n),$$

where $\Gamma(n)$ is the gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx,$$

$n > 0$. It is easy to show that

$$\begin{aligned}\int_0^{\infty} e^{tx^2} \phi(x) x^m dx &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(x^2/2)(1-2t)} x^m dx \\ &= \frac{2^{\frac{m}{2}}}{2\sqrt{\pi}} (1-2t)^{-\frac{1}{2}(m+1)} \Gamma\left(\frac{m+1}{2}\right),\end{aligned}$$

where m is an even integer. Hence,

$$\begin{aligned}
 2 \int_0^\infty e^{tx^2} \phi(x) dx &= \frac{1}{\sqrt{\pi}} (1-2t)^{-1/2} \Gamma\left(\frac{1}{2}\right) \\
 &= (1-2t)^{-1/2}, \\
 2 \int_0^\infty e^{tx^2} \phi(x) H_4(x) dx &= 2 \int_0^\infty e^{tx^2} \phi(x) (x^4 - 6x^2 + 3) dx \\
 &= \frac{4}{\sqrt{\pi}} (1-2t)^{-5/2} \Gamma\left(\frac{5}{2}\right) \\
 &\quad - \frac{12}{\sqrt{\pi}} (1-2t)^{-3/2} \Gamma\left(\frac{3}{2}\right) + 3(1-2t)^{-1/2}, \\
 2 \int_0^\infty e^{tx^2} \phi(x) H_6(x) dx &= 2 \int_0^\infty e^{tx^2} \phi(x) (x^6 - 15x^4 + 45x^2 - 15) dx \\
 &= \frac{8}{\sqrt{\pi}} (1-2t)^{-7/2} \Gamma\left(\frac{7}{2}\right) - \frac{60}{\sqrt{\pi}} (1-2t)^{-5/2} \Gamma\left(\frac{5}{2}\right) \\
 &\quad + \frac{90}{\sqrt{\pi}} (1-2t)^{-3/2} \Gamma\left(\frac{3}{2}\right) - 15(1-2t)^{-1/2}.
 \end{aligned}$$

The last three integrals can be substituted in the formula for $\psi(t)$. The moment generating function of W is $[\psi(t)]^n$. On the other hand, the moment generating function of a chi-squared distribution with n degrees of freedom is $(1-2t)^{-n/2}$ [see Example 6.9.8 concerning the moment generating function of a gamma distribution $G(\alpha, \beta)$, of which the chi-squared distribution is a special case with $\alpha = n/\alpha$, $\beta = 2$].

CHAPTER 11

11.1. (a)

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x dx = \pi, \\
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx
 \end{aligned}$$

$$= -\frac{2}{n\pi} \int_0^\pi \sin nx \, dx$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1],$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi |x| \sin nx \, dx$$

$$= 0,$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right).$$

(b)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^\pi |\sin x| \, dx$$

$$= \frac{2}{\pi} \int_0^\pi \sin x \, dx = \frac{4}{\pi},$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^\pi |\sin x| \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^\pi [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= -2 \frac{(-1)^n + 1}{\pi(n^2 - 1)}, \quad n \neq 1,$$

$$a_1 = 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi |\sin x| \sin nx \, dx$$

$$= 0,$$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots \right).$$

(c)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx$$

$$= \frac{2\pi^2}{3},$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{4}{n^2} \cos n\pi = (-1)^n \frac{4}{n^2},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= (-1)^{n-1} \frac{2}{n},$$

$$x + x^2 = \frac{\pi^2}{3} + 4 \left(-\cos x + \frac{1}{2} \sin x \right)$$

$$- 4 \left(-\frac{1}{2^2} \cos 2x + \frac{1}{4} \sin 2x \right) + \cdots$$

for $-\pi < x < \pi$. When $x = \mp \pi$, the sum of the series is $\frac{1}{2}[(-\pi + \pi^2) + (\pi + \pi^2)] = \pi^2$, that is, $\pi^2 = \pi^2/3 + 4(1 + 1/2^2 + 1/3^2 + \cdots)$.

11.2. From Example 11.2.2, we have

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Putting $x = \mp \pi$, we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Using $x = 0$, we obtain

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

or equivalently,

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Adding the two series corresponding to $\pi^2/6$ and $\pi^2/12$, we get

$$\frac{3\pi^2}{12} = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

that is,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

11.3. By Theorem 11.3.1, we have

$$f'(x) = \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx).$$

Furthermore, inequality (11.28) indicates that $\sum_{n=1}^{\infty} (n^2 b_n^2 + n^2 a_n^2)$ is a convergent series. It follows that $nb_n \rightarrow 0$ and $na_n \rightarrow 0$ as $n \rightarrow \infty$ (see Result 5.2.1 in Section 5.2).

11.4. For $m > n$, we have

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=n+1}^m (a_k \cos kx + b_k \sin kx) \right| \\ &\leq \sum_{k=n+1}^m (a_k^2 + b_k^2)^{1/2} \\ &= \sum_{k=n+1}^m \frac{1}{k} (\alpha_k^2 + \beta_k^2)^{1/2} \quad [\text{see (11.27)}] \\ &\leq \left(\sum_{k=n+1}^m \frac{1}{k^2} \right)^{1/2} \left[\sum_{k=n+1}^m (\alpha_k^2 + \beta_k^2) \right]^{1/2}. \end{aligned}$$

But, by inequality (11.28),

$$\sum_{k=n+1}^m (\alpha_k^2 + \beta_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 dx$$

and

$$\begin{aligned} \sum_{k=n+1}^m \frac{1}{k^2} &\leq \sum_{k=n+1}^{\infty} \frac{1}{k^2} \\ &\leq \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n}. \end{aligned}$$

Thus,

$$|s_m(x) - s_n(x)| \leq \frac{c}{\sqrt{n}},$$

where

$$c^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 dx.$$

By Theorem 11.3.1(b), $s_m(x) \rightarrow f(x)$ on $[-\pi, \pi]$ as $m \rightarrow \infty$. Thus, by letting $m \rightarrow \infty$, we get

$$|f(x) - s_n(x)| \leq \frac{c}{\sqrt{n}}.$$

11.5. (a) From the proof of Theorem 11.3.2, we have

$$\sum_{n=1}^{\infty} \frac{b_n}{n} \cos n\pi = \frac{A_0}{2} - \frac{a_0\pi}{2}.$$

This implies that the series $\sum_{n=1}^{\infty} (-1)^n b_n/n$ is convergent.

(b) From the proof of Theorem 11.3.2, we have

$$\int_{-\pi}^x f(t) dt = \frac{a_0(\pi+x)}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin nx - \frac{b_n}{n} (\cos nx - \cos n\pi) \right].$$

Putting $x = 0$, we get

$$\int_{-\pi}^0 f(t) dt = \frac{a_0\pi}{2} - \sum_{n=1}^{\infty} \frac{b_n}{n} [1 - (-1)^n].$$

This implies convergence of the series $\sum_{n=1}^{\infty} (b_n/n)[1 - (-1)^n]$. Since $\sum_{n=1}^{\infty} (-1)^n b_n/n$ is convergent by part (a), then so is $\sum_{n=1}^{\infty} b_n/n$.

11.6. Using the hint and part (b) of Exercise 11.5, the series $\sum_{n=2}^{\infty} b_n/n$ would be convergent, where $b_n = 1/\log n$. However, the series $\sum_{n=2}^{\infty} 1/(n \log n)$ is divergent by Maclaurin's integral test (see Theorem 6.5.4).

11.7. (a) From Example 11.2.1, we have

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

for $-\pi < x < \pi$. Hence,

$$\begin{aligned} \frac{x^2}{4} &= \int_0^x \frac{t}{2} dt \\ &= - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{4} dx \\ &= \frac{\pi^2}{12}. \end{aligned}$$

Hence,

$$\frac{x^2}{4} = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx, \quad -\pi < x < \pi.$$

(b) This follows directly from part (a).

11.8. Using the result in Exercise 11.7, we obtain

$$\begin{aligned} \int_0^x \left(\frac{\pi^2}{12} - \frac{t^2}{4} \right) dt &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_0^x \cos nt dt \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx, \quad -\pi < x < \pi. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx = \frac{\pi^2}{12} x - \frac{x^3}{12}, \quad -\pi < x < \pi.$$

11.9.

$$F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} e^{-iwx} dx,$$

$$F'(w) = \frac{i}{4\pi} \int_{-\infty}^{\infty} (-2xe^{-x^2}) e^{-iwx} dx$$

(exchanging the order of integration and differentiation is permissible here by an extension of Theorem 7.10.1). Integrating by parts, we obtain

$$\begin{aligned} F'(w) &= \frac{i}{4\pi} \left[e^{-x^2} e^{-iwx} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2} (-iwe^{-iwx}) dx \right] \\ &= \frac{-i}{4\pi} \int_{-\infty}^{\infty} e^{-x^2} (-iw) e^{-iwx} dx \\ &= -\frac{w}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} e^{-iwx} dx \\ &= -\frac{w}{2} F(w). \end{aligned}$$

The general solution of this differential equation is

$$F(w) = Ae^{-w^2/4},$$

where A is a constant. Putting $w = 0$, we obtain

$$\begin{aligned} A = F(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} dx \\ &= \frac{\sqrt{\pi}}{2\pi} = \frac{1}{2\sqrt{\pi}}. \end{aligned}$$

Hence, $F(w) = (1/2\sqrt{\pi})e^{-w^2/4}$.

11.10. Let $H(w)$ be the Fourier transform of $(f * g)(x)$. Then

$$\begin{aligned} H(w) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-y)g(y) dy \right] e^{-iwx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{\infty} f(x-y) e^{-iwx} dx \right] dy \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-y) e^{-iwx(x-y)} dx \right] g(y) e^{-iwy} dy \\ &= F(w) \int_{-\infty}^{\infty} g(y) e^{-iwy} dy \\ &= 2\pi F(w)G(w). \end{aligned}$$

11.11. Apply the results in Exercises 11.9 and 11.10 to find the Fourier transform of $f(x)$; then apply the inverse Fourier transform theorem (Theorem 11.6.3) to find $f(x)$.

11.12. (a)

$$\begin{aligned} s_n(x_n) &= \sum_{k=1}^n \frac{2(-1)^{k+1}}{k} \sin \left[k \left(\pi - \frac{\pi}{n} \right) \right] \\ &= \sum_{k=1}^n \frac{2(-1)^{k+1}}{k} (-1)^{k+1} \sin \left(\frac{k\pi}{n} \right) \\ &= \sum_{k=1}^n \frac{2 \sin(k\pi/n)}{k}. \end{aligned}$$

(b)

$$s_n(x_n) = 2 \sum_{k=1}^n \frac{\sin(k\pi/n)}{k\pi/n} \frac{\pi}{n}.$$

By dividing the interval $[0, \pi]$ into n subintervals $[(k-1)\pi/n, k\pi/n]$, $1 \leq k \leq n$, each of length π/n , it is easy to see that $s_n(x_n)$ is a Riemann sum $S(P, g)$ for the function $g(x) = (2 \sin x)/x$. Here, $g(x)$ is evaluated at the right-hand end point of each subinterval of the partition P (see Section 6.1). Thus,

$$\lim_{n \rightarrow \infty} s_n(x_n) = 2 \int_0^{\pi} \frac{\sin x}{x} dx.$$

11.13. (a) $\hat{y} = 8.512 + 3.198 \cos \phi - 0.922 \sin \phi + 1.903 \cos 2\phi + 3.017 \sin 2\phi$.

(b) Estimates of the locations of minimum and maximum resistance are $\phi = 0.7944\pi$ and $\phi = 0.1153\pi$, respectively. [Note: For more details, see Kupper (1972), Section 5.]

11.15. (a)

$$\begin{aligned} s_n^* &= \frac{1}{\sigma\sqrt{n}} \left(\sum_{j=1}^n Y_j - n\mu \right) \\ &= \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n U_j, \end{aligned}$$

where $U_j = Y_j - \mu$, $j = 1, 2, \dots, n$. Note that $E(U_j) = 0$ and $\text{Var}(U_j) = \sigma^2$. The characteristic function of s_n^* is

$$\begin{aligned}\phi_n(t) &= E(e^{it s_n^*}) \\ &= E(e^{it \sum_{j=1}^n U_j / \sigma \sqrt{n}}) \\ &= \prod_{j=1}^n E(e^{it U_j / \sigma \sqrt{n}}).\end{aligned}$$

Now,

$$\begin{aligned}E(e^{it U_j}) &= E\left(1 + it U_j - \frac{t^2}{2} U_j^2 + \dots\right) \\ &= 1 - \frac{t^2 \sigma^2}{2} + o(t^2), \\ E(e^{it U_j / \sigma \sqrt{n}}) &= 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right).\end{aligned}$$

Hence,

$$\phi_n(t) = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n.$$

Let

$$\begin{aligned}\omega_n &= -\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) \\ &= \frac{t^2}{n} \left[-\frac{1}{2} + o(1)\right], \\ \phi_n(t) &= \left[(1 + \omega_n)^{t^2/\omega_n}\right]^{-1/2 + o(1)} \\ &= \left[(1 + \omega_n)^{t^2/\omega_n}\right]^{-1/2} \left[(1 + \omega_n)^{t^2/\omega_n}\right]^{o(1)}.\end{aligned}$$

As $n \rightarrow \infty$, $\omega_n \rightarrow 0$ and

$$\begin{aligned}\left[(1 + \omega_n)^{t^2/\omega_n}\right]^{-1/2} &\rightarrow e^{-t^2/2}, \\ \left[(1 + \omega_n)^{t^2/\omega_n}\right]^{o(1)} &\rightarrow 1.\end{aligned}$$

Thus,

$$\phi_n(t) \rightarrow e^{-t^2/2} \quad \text{as } n \rightarrow \infty.$$

(b) $e^{-t^2/2}$ is the characteristic function of the standard normal distribution Z . Hence, by Theorem 11.7.3, as $n \rightarrow \infty$, $s_n^* \rightarrow Z$.

CHAPTER 12

12.1. (a)

$$\begin{aligned} I_n &\approx S_n \\ &= \frac{h}{2} \left(\log 1 + \log n + 2 \sum_{i=1}^{n-1} \log x_i \right) \\ &= \frac{n-1}{2(n-1)} \left(\log n + 2 \sum_{i=2}^{n-1} \log i \right) \\ &= \frac{1}{2} \log n + \log 2 + \log 3 + \cdots + \log(n-1) \\ &= \frac{1}{2} \log n + \log(n!) - \log n \\ &= \log(n!) - \frac{1}{2} \log n. \end{aligned}$$

(b) $n \log n - n + 1 \approx \log(n!) - \frac{1}{2} \log n$. Hence,

$$\begin{aligned} \log(n!) &\approx \left(n + \frac{1}{2}\right) \log n - n + 1 \\ n! &\approx \exp\left[\left(n + \frac{1}{2}\right) \log n - n + 1\right] \\ &= e e^{-n} n^{n+1/2} = e^{-n+1} n^{n+1/2}. \end{aligned}$$

12.2. Applying formula (12.10), we obtain the following approximate values for the integral $\int_0^1 dx/(1+x)$:

n	Approximation
2	0.69325395
4	0.69315450
8	0.69314759
16	0.69314708

The exact value of the integral is $\log 2 = 0.69314718$. Using formula (12.12), the error of approximation is less than or equal to

$$\frac{M_4(b-a)^5}{2880n^4} = \frac{M_4}{2880(8)^4},$$

where M_4 is an upper bound on $|f^{(4)}(x)|$ for $0 \leq x \leq 1$. Here, $f(x) = 1/(1+x)$, and

$$f^{(4)}(x) = \frac{24}{(1+x)^5}.$$

Hence, $M_4 = 24$, and

$$\frac{M_4}{2880(8)^4} \approx 0.000002.$$

12.3. Let

$$z = \frac{2\theta - \pi/2}{\pi/2}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

$$\theta = \frac{\pi}{4}(1+z), \quad -1 \leq z \leq 1.$$

Then

$$\begin{aligned} \int_0^{\pi/2} \sin \theta \, d\theta &= \frac{\pi}{4} \int_{-1}^1 \sin \left[\frac{\pi}{4}(1+z) \right] dz \\ &\approx \frac{\pi}{4} \sum_{i=0}^2 \omega_i \sin \left[\frac{\pi}{4}(1+z_i) \right]. \end{aligned}$$

Using the information in Example 12.4.1, we have $z_0 = -\sqrt{\frac{3}{5}}$, $z_1 = 0$, $z_2 = \sqrt{\frac{3}{5}}$; $\omega_0 = \frac{5}{9}$, $\omega_1 = \frac{8}{9}$, $\omega_2 = \frac{5}{9}$. Hence,

$$\begin{aligned} \int_0^{\pi/2} \sin \theta \, d\theta &\approx \frac{\pi}{4} \left\{ \frac{5}{9} \sin \left[\frac{\pi}{4} \left(1 - \sqrt{\frac{3}{5}} \right) \right] + \frac{8}{9} \sin \frac{\pi}{4} \right. \\ &\quad \left. + \frac{5}{9} \sin \left[\frac{\pi}{4} \left(1 + \sqrt{\frac{3}{5}} \right) \right] \right\} \\ &= 1.0000081. \end{aligned}$$

The error of approximation associated with $\int_{-1}^1 \sin[(\pi/4)(1+z)] dz$ is obtained by applying formula (12.16) with $a = -1$, $b = 1$, $f(z) = \sin[(\pi/4)(1+z)]$, $n = 2$. Thus, $f^{(6)}(\xi) = -(\pi/4)^6 \sin[(\pi/4)(1+\xi)]$, $-1 < \xi < 1$, and the error is therefore less than

$$\frac{2^7(3!)^4}{7(6!)^3} \left(\frac{\pi}{4} \right)^6.$$

Consequently, the error associated with $\int_0^{\pi/2} \sin \theta \, d\theta$ is less than

$$\begin{aligned} \frac{\pi}{4} \frac{2^7 [3!]^4}{7[6!]^3} \left(\frac{\pi}{4}\right)^6 &= \left(\frac{\pi}{2}\right)^7 \frac{[3!]^4}{7[6!]^3} \\ &= 0.0000117. \end{aligned}$$

12.4. (a) The inequality is obvious from the hint.

(b) Let $m = 3$. Then,

$$\begin{aligned} \frac{1}{m} e^{-m^2} &= \frac{1}{3} e^{-9} \\ &= 0.0000411. \end{aligned}$$

(c) Apply Simpson's method on $[0, 3]$ with $h < 0.2$. Here,

$$f^{(4)}(x) = (12 - 48x^2 + 16x^4)e^{-x^2},$$

with a maximum of 12 at $x = 0$ over $[0, 3]$. Hence, $M_4 = 12$. Using the fact that $h = (b - a)/2n$, we get $n = (b - a)/2h > 3/0.4 = 7.5$. Formula (12.12), with $n = 8$, gives

$$\begin{aligned} \frac{nM_4 h^5}{90} &< \frac{8(12)(0.2)^5}{90} \\ &= 0.00034. \end{aligned}$$

Hence, the total error of approximation for $\int_0^\infty e^{-x^2} dx$ is less than $0.0000411 + 0.00034 = 0.00038$. This makes the approximation correct to three decimal places.

12.5.

$$\begin{aligned} \int_0^\infty \frac{x}{e^x + e^{-x} - 1} dx &= \int_0^\infty \frac{x e^{-x}}{1 + e^{-2x} - e^{-x}} dx, \\ &\approx \sum_{i=0}^n \omega_i \frac{x_i}{1 + e^{-2x_i} - e^{-x_i}}, \end{aligned}$$

where the x_i 's are the zeros of the Laguerre polynomial $L_{n+1}(x)$. Using the entries in Table 12.1 for $n = 1$, we get

$$\begin{aligned} \int_0^\infty \frac{x}{e^x + e^{-x} - 1} dx &\approx 0.85355 \frac{0.58579}{1 + e^{-2(0.58579)} - e^{-0.58579}} \\ &\quad + 0.14645 \frac{3.41421}{1 + e^{-2(3.41421)} - e^{-3.41421}} = 1.18. \end{aligned}$$

12.6. (a) Let $u = (2t - x)/x$. Then

$$\begin{aligned} I(x) &= \frac{1}{2} \int_{-1}^1 \frac{x}{1 + \frac{1}{4}x^2(u+1)^2} du \\ &= 2x \int_{-1}^1 \frac{1}{4 + x^2(u+1)^2} du. \end{aligned}$$

(b) Applying a Gauss–Legendre approximation of $I(x)$ with $n = 4$, we obtain

$$I(x) \approx 2x \sum_{i=0}^4 \frac{\omega_i}{4 + x^2(u_i + 1)^2},$$

where u_0, u_1, u_2, u_3 , and u_4 are the five zeros of the Legendre polynomial $P_5(u)$ of degree 5. Values of these zeros and the corresponding ω_i 's are given below:

$$\begin{aligned} u_0 &= 0.90617985, & \omega_0 &= 0.23692689, \\ u_1 &= 0.53846931, & \omega_1 &= 0.47862867, \\ u_2 &= 0, & \omega_2 &= 0.56888889, \\ u_3 &= -0.53846931, & \omega_3 &= 0.47862867, \\ u_4 &= -0.90617985, & \omega_4 &= 0.23692689, \end{aligned}$$

(see Table 7.5 in Shoup, 1984, page 202).

12.7. $\int_0^1 (\cos x)^\lambda dx = \int_0^1 e^{\lambda \log \cos x} dx$. Using the method of Laplace, the function $h(x) = \log \cos x$ has a single maximum in $[0, 1]$ at $x = 0$. Formula (12.28) gives

$$\int_0^1 e^{\lambda \log \cos x} dx \sim \left[\frac{-\pi}{2\lambda h''(0)} \right]^{1/2}$$

as $\lambda \rightarrow \infty$, where

$$\begin{aligned} h''(0) &= - \frac{1}{\cos^2 x} \Big|_{x=0} \\ &= -1. \end{aligned}$$

Hence,

$$\int_0^1 e^{\lambda \log \cos x} dx \sim \sqrt{\frac{\pi}{2\lambda}}.$$

12.9.

$$\int_0^1 \int_0^{(1-x_1^2)^{1/2}} (1-x_1^2-x_2^2)^{1/2} dx_1 dx_2 = \frac{\pi}{6} \\ \approx 0.52359878.$$

Applying the 16-point Gauss–Legendre rule to both formulas gives the approximate value 0.52362038.

[Note: For more details, see Stroud (1971, Section 1.6).]

12.10.

$$\int_0^\infty (1+x^2)^{-n} dx = \int_0^\infty e^{-n \log(1+x^2)} dx$$

Apply the method of Laplace with $\varphi(x) = 1$ and

$$h(x) = -\log(1+x^2).$$

This function has a single maximum in $[0, \infty)$ at $x = 0$. Using formula (12.28), we get

$$\int_0^\infty (1+x^2)^{-n} dx \sim \left[\frac{-\pi}{2nh''(0)} \right]^{1/2},$$

where $h''(0) = -2$. Hence,

$$\int_0^\infty (1+x^2)^{-n} dx \sim \sqrt{\frac{\pi}{4n}} = \frac{1}{2} \sqrt{\frac{\pi}{n}}.$$

12.11.

Number of Quadrature Points	Approximate Value of Variance
2	0.1175
4	0.2380
8	0.2341
16	0.1612
32	0.1379
64	0.1372

[Source: Example 16.6.1 in Lange (1999).]

12.12. If $x \geq 0$, then

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt. \end{aligned}$$

Using Maclaurin's series (see Section 4.3), we obtain

$$\begin{aligned}
 \Phi(x) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x \left[1 + \frac{1}{1!} \left(-\frac{t^2}{2} \right) + \frac{1}{2!} \left(-\frac{t^2}{2} \right)^2 \right. \\
 &\quad \left. + \frac{1}{3!} \left(-\frac{t^2}{2} \right)^3 + \cdots + \frac{1}{n!} \left(-\frac{t^2}{2} \right)^n + \cdots \right] dt \\
 &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left[t - \frac{t^3}{3 \times 2 \times 1!} + \frac{t^5}{5 \times 2^2 \times 2!} - \frac{t^7}{7 \times 2^3 \times 3!} \right. \\
 &\quad \left. + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1) \times 2^n \times n!} + \cdots \right]_0^x \\
 &= \frac{1}{2} + \frac{x}{\sqrt{2\pi}} \left[1 - \frac{x^2}{3 \times 2 \times 1!} + \frac{x^4}{5 \times 2^2 \times 2!} - \frac{x^6}{7 \times 2^3 \times 3!} \right. \\
 &\quad \left. + \cdots + (-1)^n \frac{x^{2n}}{(2n+1) \times 2^n \times n!} + \cdots \right].
 \end{aligned}$$

12.13. From Exercise 12.12 we need to find values of a, b, c, d such that

$$1 + ax^2 + bx^4 \approx (1 + cx^2 + dx^4) \left(1 - \frac{x^2}{6} + \frac{x^4}{40} - \frac{x^6}{336} + \frac{x^8}{3456} \right).$$

Equating the coefficients of x^2, x^4, x^6, x^8 on both sides yields four equations in a, b, c, d . Solving these equations, we get

$$a = \frac{17}{468}, \quad b = \frac{739}{196560}, \quad c = \frac{95}{468}, \quad d = \frac{55}{4368}.$$

[*Note:* For more details, see Morland (1998).]

12.14.

$$\begin{aligned}
 y = G(x) &= \int_0^x \frac{1}{4} (1+t) dt \\
 &= \frac{1}{4} \left(x + \frac{x^2}{2} \right), \quad 0 \leq x \leq 2.
 \end{aligned}$$

The only solution of $y = G(x)$ in $[0, 2]$ is $x = -1 + (1 + 8y)^{1/2}$, $0 \leq y$

≤ 1 . A random sample of size 150 from the $g(x)$ distribution is given by

$$x_i = -1 + (1 + 8y_i)^{1/2}, \quad i = 1, 2, \dots, 150,$$

where the y_i 's form a random sample from the $U(0, 1)$ distribution. Applying formula (12.43), we get

$$\begin{aligned} \hat{I}_{150}^* &= \frac{4}{150} \sum_{i=1}^{150} \frac{e^{x_i^2}}{1 + x_i} \\ &= 16.6572. \end{aligned}$$

Using now formula (12.36), we obtain

$$\begin{aligned} \hat{I}_{150} &= \frac{2}{150} \sum_{i=1}^{150} e^{u_i^2} \\ &= 17.8878, \end{aligned}$$

where the u_i 's form a random sample from the $U(0, 2)$ distribution.

12.15. As $n \rightarrow \infty$,

$$\begin{aligned} \left[1 + \frac{x^2}{n}\right]^{-(n+1)/2} &= \left[1 + \frac{x^2}{n}\right]^{-n/2} \left[1 + \frac{x^2}{n}\right]^{-1/2} \\ &\sim e^{-x^2/2}. \end{aligned}$$

Furthermore, by applying Stirling's formula (formula 12.31), we obtain

$$\begin{aligned} \Gamma\left(\frac{n+1}{2}\right) &\sim e^{-\frac{1}{2}(n-1)} \left[\frac{n-1}{2}\right]^{(n-1)/2} \left[2\pi\left(\frac{n-1}{2}\right)\right]^{1/2}, \\ \Gamma\left(\frac{n}{2}\right) &\sim e^{-\frac{1}{2}(n-2)} \left[\frac{n-2}{2}\right]^{(n-2)/2} \left[2\pi\left(\frac{n-2}{2}\right)\right]^{1/2}. \end{aligned}$$

Hence,

$$\begin{aligned}
 \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} &\sim \frac{1}{\sqrt{n\pi}} \frac{1}{e^{1/2}} \left[\frac{n-1}{n-2}\right]^{n/2} \frac{n-2}{2\sqrt{\frac{n-1}{2}}} \left[\frac{n-1}{n-2}\right]^{1/2} \\
 &\sim \frac{1}{\sqrt{n\pi}} \frac{1}{e^{1/2}} \frac{\left(1 - \frac{1}{n}\right)^{n/2}}{\left(1 - \frac{2}{n}\right)^{n/2}} \frac{n-2}{2\sqrt{\frac{n-1}{2}}} \\
 &\sim \frac{1}{\sqrt{n\pi}} \frac{1}{e^{1/2}} \frac{e^{-1/2}}{e^{-1}} \sqrt{\frac{n}{2}} \\
 &= \frac{1}{\sqrt{2\pi}}.
 \end{aligned}$$

Hence, for large n ,

$$F(x) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$